J. Fluid Mech. (1984), *vol.* 143, *pp.* 197–222
Printed in Great Britain

Third sound: the propagation of waves on the surface of superfluid helium with healing and relaxation

By R. S. JOHNSON

School of Mathematics, The University, Newcastle upon Tyne, NE1 7RU, U.K.

(Received 26 October 1983)

The propagation of surface waves – that is 'third' sound – on superfluid helium is considered. The fluid is treated as a continuum, using the two-fluid model of Landau, and incorporating the effects of healing, relaxation, thermal conductivity and Newtonian viscosity. Under the assumptions only of a small amplitude and long wavelength, a linear theory is developed which includes some discussion of the matching to the outer regions of the vapour. This results in a comprehensive propagation speed for linear waves, although a few properties of the flow are left undetermined at this order. A nonlinear theory is then outlined (without dwelling on the details) which leads to the Burgers equation in an appropriate far field, and enables the leading-order theory to be concluded.

Some numerical results, for two temperatures, are presented by first recording the Helmholtz free energy as a polynomial in densities. Owing to the inaccuracies inherent in this procedure, only the equilibrium state can be satisfactorily reproduced, but this is sufficient to predict the onset phenomenon. The propagation speed, as a function of film thickness, is roughly estimated by using the earlier results of Johnson (without healing) suitably doctored to incorporate the new computed superfluid density distributions. The looked-for reduction in the predicted speeds is evident, but the magnitude of this reduction is too large for very thin films. However, it is hoped that the analytical results presented here will prove more effective when a complete and accurate description of the Helmholtz free energy is available.

1. Introduction

Superfluid helium (or 'helium II') is a quantum fluid which is produced when liquid helium is cooled below the λ-point (a second-order phase transition at about 2.2 K). Virtually all the known properties of helium II can be explained with some considerable degree of success by representing the fluid as a mixture of two components: the so-called two-fluid model of Landau (Landau 1941; Landau & Lifshitz 1959; Hills & Roberts 1972). The two components are taken to be the 'superfluid', which moves irrotationally, and the 'normal fluid', which is a classical Newtonian viscous fluid. The helium vapour is usually treated as a viscous compressible gas (when the vapour is modelled at all), and this author will adopt the same view. We thus have the basis for a continuum theory describing the motion of superfluid helium.

There are quite a number of different types of disturbance that can propagate in helium II. These may involve a motion that is predominantly associated with the superfluid component, or with some appropriate coupling between the components.

(For a general background, and a discussion of the simpler modes of propagation, see Putterman (1974).) Each mode of propagation is given a number, and third sound is the designation of the disturbance on the surface of liquid helium II. Third sound is therefore the analogue of the classical water-wave problem, although here the body-force potential is dominated by (or is solely given by) the van der Waals term. Of course, as might well be anticipated, the corresponding analysis for helium II is vastly more involved than that for water waves, and not only by virtue of the two-component character. The superfluid helium in general, and third sound in particular, enjoy a number of complicating features. Helium II itself requires a model which accommodates the phenomenon of 'healing', wherein the superfluid density drops to zero at an infinite potential barrier (such as a solid wall), and it is generally accepted that the same pertains at a free surface. For further discussion of this point see Sobaynin (1972) and Ginzburg & Sobaynin (1976). The governing equations must also include a constitutive relation which describes the conversion of superfluid to normal fluid (and *vice versa*), and which is, furthermore, a relaxation process. (In the equilibrium state there is only normal fluid at the λ-point, and only superfluid at absolute zero.) The appropriate extension of Landau's two-fluid equations to incorporate both healing and relaxation is given by Hills & Roberts (1977).

At the free surface, apart from the usual kinematic and stress conditions (suitably amended to account for the effects of evaporation/condensation and healing), we also require an energy balance equation. This relates the heat transfer and latent heat to the difference in chemical potential across the surface. The temperature is, of course, to be continuous at the interface between vapour and liquid. All the relevant interfacial conditions between helium II and its vapour are derived in Hills & Roberts (1979).

Third sound was first predicted by Atkins (1959) and then detected by Everitt, Atkins & Denenstein in 1962, albeit on rather thick films of helium II (up to about 10 cm). We are concerned here, in the main, with the thinner films in the range 5–100 Å (1 Å = 10^{-10} m); this corresponds to a thickness of not more than about 25 atomic layers (see Atkins & Rudnick 1970). The emphasis on thin films might seem to be at variance with the continuum approach that we are pursuing, but our interest is, after all, centred on very long waves and it is one of the surprises that a macroscopic theory can cope so well with what is essentially a quantum – and therefore microscopic – phenomenon. In fact it will be shown that some success is achieved at thicknesses as little as $1\frac{1}{2}$ atomic layers. One of the purposes behind this analysis is to demonstrate the wealth of detail that is available in a suitably complete continuum theory. We would argue, therefore, that there is some virtue in delineating any limitations of these governing equations before rejecting them wholesale in favour of a fully microscopic theory. We cannot expect our equations to be an infallible representation of helium II in general, although, of course, our concern here is only with those questions that might arise in the description of third sound.

Elementary theories of third sound, based on simplified governing equations which describe the essential character of the problem, are given by Atkins & Rudnick (1970) and Putterman (1974). A more detailed representation of the liquid and vapour – and its coupling – is to be found in Bergman (1969, 1971). However, this latter author does not make use of asymptotic methods, but rather he develops a solution based on *ad hoc* approximations: healing and relaxation are omitted and the equilibrium state is absorbed into average properties across the film. In an effort to present a more systematic theory, Johnson (1978*a*) constructed an asymptotic solution for the linearized problem describing small-amplitude long waves. Nevertheless it proved expedient to describe the coupling to the vapour by a set of idealized interfacial conditions, and only compressibility of the liquid was retained in the numerical results presented for the propagation speed. Recently, Johnson (1984) has given a more complete specification of all the governing equations (for third sound), together with a brief discussion of various aspects of the problem. In the case of the one analysis – a linear theory – in which healing was retained, again the idealized surface conditions were employed, and so no detailed model of the vapour was utilized. The primary aim of this current work is therefore to remedy some of these shortcomings and endeavour to present a fairly comprehensive account of the linearized problem for third sound. The only simplifying assumptions will be those consistent with the propagation of small-amplitude long waves. Furthermore, it will also be possible to touch on the form of the nonlinear far field as well as the problem of matching to the vapour.

We shall take the opportunity to present some numerical results, although these will be far from complete. At the present there is not readily available a wholly satisfactory model for the Helmholtz free energy A which can also be converted into a numerical form: this is necessary if the ultimate aim is to produce worthwhile comparisons with experimental data as well as reliable predictions. Consequently certain features of the problem must, perforce, be left in an analytical form (which in some cases will mean an ordinary differential equation, since some coefficients of these equations depend on A and, in particular, derivatives of A).

The philosophy adopted here, being a continuation of that developed in Johnson (1984), is to construct the appropriate non-dimensional equations so that the parameters describing amplitude and wavelength appear explicitly. This is quite straightforward if all the physical coefficients are non-dimensionalized using the depth h of the film as the lengthscale: we treat h as fixed for a particular flow configuration, which makes good sense from the experimental standpoint. Introducing a typical wavelength λ as the lengthscale along the film, we then obtain ϵ ($=$ typical amplitude/h) and δ ($= h/\lambda$) as the only parameters incorporating amplitude and wavelength. The problem of small-amplitude long waves is therefore obtained from $\epsilon \to 0$, $\delta \to 0$, whilst keeping all the other parameters fixed. The additional parameters will therefore include those associated with viscosity, thermal conductivity, healing, etc. (It should be noted that we may wish to examine special classes of problem for which $\epsilon \to 0$, $\delta \to 0$, but such that $\delta = \delta(\epsilon)$ is specified.)

It is out of place to recapitulate the background to the governing equations: the formulation and non-dimensionalization relevant to third sound are to be found in Johnson (1984; 1978*a* also, although less complete in this context). We shall, however, describe the notation employed and comment on one or two of the parameters that are introduced; this should be sufficient to make the equations fairly comprehensible. The full set of non-dimensional equations is reproduced in Appendix A.

2. Equations and notation: a résumé

We have already commented on the choice of the depth h of the liquid helium layer as the principal lengthscale. Only the coordinate along the film, the time and the velocity potential for the superfluid are non-dimensionalized using a typical wavelength λ. In addition we require a typical speed of the wave (c', say) to complete the non-dimensionalization of the kinematic quantities. We may now begin by recording the coordinate system and other notation that is required. (All the variables

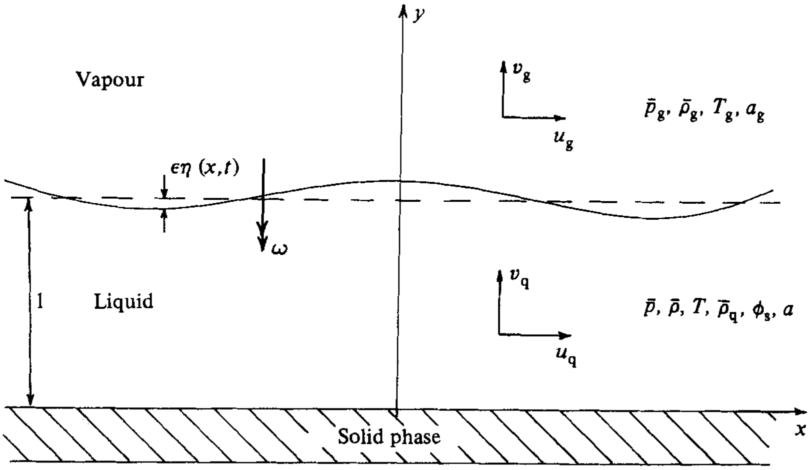

FIGURE 1. Defining sketch for flow configuration and variables.

introduced hereinafter are non-dimensional, unless stated otherwise.) The rectangular Cartesian coordinates are (x, y), where y is measured across the film from the substrate (a solid layer) to the free surface, and t is the time. The velocity components are written as (u_q, v_q), where $q \equiv \mathrm{s, n}$ denotes either the superfluid or the normal fluid. The surface perturbation is η, and ϕ_s is the velocity potential. For the non-dimensionalization of the thermodynamic quantities we require a typical temperature (T_0, the equilibrium temperature) and an average total density of the liquid ρ_0. (It should be noted that the liquid is not assumed incompressible: the effects of compressibility cannot be ignored *a priori* since there is a large variation of pressure across the film, from about 0 to 25 atm; 1 atm $\approx$ 101 kPa.) We are now able to define the total density $\bar{\rho}$, the component densities $\bar{\rho}_q$ ($q \equiv \mathrm{n, s}$), pressure $\bar{p}$ and temperature T. The Helmholtz free energy is written as a and the body force potential is ω.

The interface conditions require the chemical potential, mass flux across the surface, and the ratio of these two, to be defined in a suitable non-dimensional form. This yields ψ, j and ν respectively, where the equation $j = \nu\psi$ (see (A 20)) is the chosen constitutive relation between ψ and j.

The helium vapour, denoted by the subscript g (for 'gas'), can be described by non-dimensional variables as outlined above, with the exception that the average density is taken to be the corresponding vapour density ρ_{g0} and the typical speed is c_g'. Nevertheless the time is still non-dimensionalized with respect to c'. Thus we are able to define (u_g, v_g), $\bar{\rho}_g, \bar{p}_g, T_g$ and a_g just as for the liquid. Now, since the vapour and liquid variables are defined using different densities and speeds, ratios of these two quantities will appear in the governing equations. However, no limiting process is associated with these additional parameters ($\sigma = \rho_{g0}/\rho_0$; $\Delta = c_g'/c'$), although they can be used to advantage if numerical estimates are desired from a given order in the asymptotic solution. The majority of these variables will be found on the defining sketch (figure 1).

The remaining physical parameters usually required for the description of a liquid and vapour, expressed in non-dimensional form, are: κ, κ_g (thermal conductivities); α_i, α_{gi} (inverse Reynolds numbers; $i = 1, 2$ for the two viscosities). The properties peculiar to helium II also have their associated parameters. If the healing coefficient is taken to be proportional to $1/\rho_s$ and that for the relaxation proportional to $\rho_s^{-\frac{1}{2}}$

(see Khalatnikov 1970; Hills & Roberts 1977), then the parameters are β and γ respectively.

The final stage, before we are in a position to present the equations themselves, involves the appreciation that the surface wave disturbs a quite intricate equilibrium state. For both the vapour and the liquid this state is a function of y, with the exception of the temperature, for which $T = T_{\mathrm{g}} = 1$ in equilibrium. The state of thermodynamic equilibrium is denoted by a circumflex, so that, in the absence of any disturbances, $\bar{p} = \hat{p}(y)$, etc. If we now introduce the appropriate perturbation of this state, we may write

$$T = 1+\epsilon\delta\tau, \quad T_{\mathrm{g}} = 1+\epsilon\delta\tau_{\mathrm{g}}, \quad \bar{p} = \hat{p}+\epsilon p, \tag{1}$$

$$\rho = \hat{\rho}+\epsilon r, \quad \bar{p}_{\mathrm{g}} = \hat{p}_{\mathrm{g}}+p_{\mathrm{g}}, \quad \bar{\rho}_q = \hat{\rho}_q+\epsilon r_q,$$

where $q \equiv \mathrm{s, n, g}$. These definitions are consistent with the governing equations, which, with this notation, appear in Appendix A.

The equilibrium state is described by the equations recovered by setting the perturbation terms to zero; this yields

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}y} = -\hat{\rho}\frac{\mathrm{d}\omega}{\mathrm{d}y}-\beta\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{1}{\hat{\rho}_{\mathrm{s}}}\left(\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}y}\right)^2\right], \quad \hat{a}+\hat{\rho}\frac{\partial\hat{a}}{\partial\hat{\rho}}+\omega(y) = 0, \tag{2a, b}$$

$$\hat{p} = \hat{\rho}^2\frac{\partial\hat{a}}{\partial\hat{\rho}}+\hat{\rho}\hat{\rho}_{\mathrm{s}}\frac{\partial\hat{a}}{\partial\hat{\rho}_{\mathrm{s}}}-\beta\frac{\mathrm{d}^2\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y^2}, \quad \hat{\rho}\frac{\partial\hat{a}}{\partial\hat{\rho}_{\mathrm{s}}}-\frac{\beta}{\hat{\rho}_{\mathrm{s}}}\frac{\mathrm{d}^2\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y^2}+\tfrac{1}{2}\beta\left(\frac{1}{\hat{\rho}_{\mathrm{s}}}\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}\right)^2 = 0$$

for the liquid, where $\hat{a} = a(\hat{\rho}, \hat{\rho}_{\mathrm{s}}, 1)$. (The Helmholtz free energy is usually expressed as a function of the total density, the superfluid density and temperature.) It is clear that the equations (2) are not independent: for example, the first three imply the fourth upon the elimination of $\hat{p}$ and ω. The vapour in equilibrium satisfies the equations

$$\Delta^2\frac{\mathrm{d}\hat{p}_{\mathrm{g}}}{\mathrm{d}y} = -\hat{\rho}_{\mathrm{g}}\frac{\mathrm{d}\omega}{\mathrm{d}y}, \quad \hat{p}_{\mathrm{g}} = \hat{\rho}_{\mathrm{g}}^2\frac{\partial\hat{a}_{\mathrm{g}}}{\partial\hat{\rho}_{\mathrm{g}}}, \tag{3a, b}$$

where $\hat{a}_{\mathrm{g}} = a_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}, 1)$, and at the interface we have

$$\left.\begin{aligned} &\hat{p}-\sigma\Delta^2\hat{p}_{\mathrm{g}}+\frac{\beta}{\hat{\rho}_{\mathrm{s}}}\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}\right)^2 = 0, && (4a)\\ &\hat{a}+\frac{\hat{p}}{\hat{\rho}} = \Delta^2\left(\hat{a}_{\mathrm{g}}+\frac{\hat{p}_{\mathrm{g}}}{\hat{\rho}_{\mathrm{g}}}\right)-\frac{\beta}{2\hat{\rho}\hat{\rho}_{\mathrm{s}}}\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}\right)^2 && (4b)\end{aligned}\right\} \text{ on } \quad y = 1.$$

The boundary conditions that we must impose are that

$$\hat{\rho}_{\mathrm{s}} = 0 \quad \text{on} \quad y = 0, 1; \quad \hat{p} = \hat{p}_{\mathrm{b}} \quad \text{on} \quad y = 0, \tag{5}$$

where $\hat{p}_{\mathrm{b}}$ is the (non-dimensional) pressure at which helium II solidifies (at the temperature $T = 1$). In passing it should be noted that (2*b*) serves to fix the arbitrary constant in the definition of $\hat{a}$, as does (4*b*) for $\hat{a}_{\mathrm{g}}$, given $\omega(y)$.

The conditions that pertain far from the surface and into the vapour ($y \rightarrow \infty$) depend critically on the choice of $\omega(y)$. If $\omega(y)$ is represented solely by the term arising from the van der Waals force of attraction towards a solid boundary, then ω takes the form

$$\frac{k_1}{(y+d)^3}[1+k_2(y+d)]^{-1},$$

where k_1, k_2 are constants and d is chosen so that solidification (i.e. $\hat{p} = \hat{p}_{\mathrm{b}}$) occurs on $y = 0$. Now, since $\omega \rightarrow 0$ as $y \rightarrow \infty$, the vapour will approach a constant pressure and density far from the surface: this seems most appropriate in view of the usual experimental configuration. On the other hand, if $\omega(y)$ were to incorporate a gravitational contribution then the pressure and density would decrease upwards (if the film were condensed onto a horizontal surface), but this would be significant only at vast distances above the surface. We shall restrict our study to the case in which ω is independent of gravity, and hence the flat surface on which the helium II is condensed may take any orientation. In the light of the comments above it is perhaps worth remembering that y is scaled on h, whence $y \to \infty$ is to be interpreted as large with respect to the thickness of the film.

An asymptotic solution is constructed which is valid as $\epsilon \to 0$ and $\delta \to 0$, although the precise ordering of the terms in the expansions cannot be determined unless a choice is made for $\delta = \delta(\epsilon)$. Nevertheless, insofar as the underlying linearized problem is concerned, there is no necessity for imposing any restriction: the limiting process $\epsilon \to 0$, $\delta \to 0$ – otherwise unspecified – will generate the (leading-order) equations relevant to small-amplitude long waves. If higher-order terms are required, and in particular if the nonlinear far field is to be examined, then $\delta = \delta(\epsilon)$ must be fixed in order to limit the class of problems under discussion. Since the amplitude is $O(\epsilon)$ we can anticipate that the nonlinear far field will be evident on a timescale $O(\epsilon^{-1})$, and then the dominant effects associated with the long-wave parameter δ will arise at this same order if we set $\delta = \epsilon$. In fact the parameter δ plays a somewhat spurious role; it has arisen solely because we have opted to use two lengthscales, albeit on sound physical principles. For, if we introduce the transformation $x \to \delta x/\epsilon$ (and correspondingly for $t, v_q, \tau, \tau_g, \phi_s$) then, in essence, ϵ replaces δ for any δ. This property of the equations may be usefully interpreted as showing that, for arbitrary δ, there exists a lengthscale on which the dominant effects associated with δ can be balanced against the nonlinear effects. (The terms involving δ presumably generate dissipative and/or dispersive effects; the precise nature of these contributions will be made clear in §4.) We shall regard the special choice $\delta = \epsilon$ as the most general problem associated with the limit process $\epsilon \to 0$, $\delta \to 0$. That this special case is sufficient will be argued in due course, so that the other possibilities ($\delta = o(\epsilon)$, $\epsilon = o(\delta)$) are, in all essentials, embedded in this one case.

The procedure we adopt is to seek an asymptotic solution as $\epsilon \to 0$ (with $\delta = \epsilon$), keeping all the other parameters fixed. In the first instance this will lead to a near-field (linear) theory, and then, by introducing suitable far-field variables, the nonlinear problem may be examined. Furthermore, it will also be possible to see how the solution for the liquid (and the vapour near the surface) may be matched to the vapour far from the surface. It is to be expected that the solution for $y = O(1)$ will not be uniformly valid as $y \to \infty$, since the evaporation/condensation at the surface requires a recirculation of the flow for large y. For simplicity we shall construct a solution valid in the liquid (and in the vapour for $y = O(1)$), for both linear and nonlinear waves, in the most obvious manner, e.g. in integer powers of ϵ. We shall therefore presuppose that there does exist an appropriate solution as $y \to \infty$ which matches the linear and nonlinear theory. The form that this matching must take, at least to leading order, will be briefly examined in order to afford some check on the validity of the assumption of existence.

3. Linear theory

A wave theory, certainly valid for $y = O(1)$, is obtained by seeking asymptotic expansions for all the dependent variables occurring in the governing equations. It is sufficient, in the initial stages at least, to assume that, for x, t and y of $O(1)$, the variables may be written in the form

$$Q = Q_0 + o(1) \quad \text{as} \quad \epsilon \to 0, \tag{6}$$

where Q represents any one of η, τ, v_s etc. However, when it becomes necessary, we shall assume that (6) represents the first term of an asymptotic expansion based on the sequence $\{\epsilon^n\}$. The functions Q_0 depend on x, t and y or just x and t, the difference arising where the functions are defined only at the surface, but there is one exception: by virtue of the scalings consistent with small-amplitude long waves we have

$$\phi_s(x, t, y; \epsilon) = \theta_s(x, t; \epsilon) + O(\epsilon^2), \tag{7}$$

whence $\theta_{s0} = \theta_{s0}(x, t)$.

The linear equations are, for completeness, given below; the zero subscript associated with the leading order is, however, omitted for ease of presentation (although it will be retained elsewhere in the text to avoid any confusion). Reference to the appropriate equation in Appendix A appears to the left. For the liquid we obtain

$$\text{(A 1)} \quad \frac{\partial r}{\partial t} + \frac{\partial}{\partial x}(u_n \hat{\rho}_n + u_s \hat{\rho}_s) + \frac{\partial}{\partial y}(v_n \hat{\rho}_n + v_s \hat{\rho}_s) = 0, \tag{8}$$

$$\text{(A 2)} \quad \frac{\partial^2 u_n}{\partial y^2} = 0, \tag{9}$$

$$\text{(A 3)} \quad \frac{\partial p}{\partial y} + r\omega' + \beta \frac{\partial}{\partial y}\left[\hat{\rho}_s'\left(2\frac{\partial r_s}{\partial y} - \frac{\hat{\rho}_s' r_s}{\hat{\rho}_s}\right)\middle/\hat{\rho}_s\right] = 0, \tag{10}$$

$$\text{(A 4)} \quad (2\hat{a}_\rho + \hat{\rho}\hat{a}_{\rho\rho})\, r + (\hat{a}_{\rho s} + \hat{\rho}\hat{a}_{\rho\rho s})\, r_s + \frac{\partial \theta_s}{\partial t} = 0, \tag{11}$$

$$\text{(A 5)} \quad p = (2\hat{\rho}\hat{a}_\rho + \hat{\rho}^2\hat{a}_{\rho\rho} + \hat{\rho}_s\,\hat{a}_{\rho s} + \hat{\rho}\hat{\rho}_s\,\hat{a}_{\rho\rho s})\, r + \hat{\rho}(\hat{a}_{\rho s} + \hat{\rho}\hat{a}_{\rho\rho s} + \hat{\rho}_s\,\hat{a}_{\rho s\rho s})\, r_s - \beta\frac{\partial^2 r_s}{\partial y^2}, \tag{12}$$

$$\text{(A 6)} \quad (\hat{a}_{\rho s} + \hat{\rho}\hat{a}_{\rho\rho s})\, r + \hat{\rho}\hat{a}_{\rho s\rho s}\, r_s + \beta\left[\hat{\rho}_s'' r_s + \hat{\rho}_s'\left(\frac{\partial r_s}{\partial y} - \frac{\hat{\rho}_s' r_s}{\hat{\rho}}\right) - \hat{\rho}_s\frac{\partial^2 r_s}{\partial y^2}\right]\middle/(\hat{\rho}_s)^2 = 0, \tag{13}$$

$$\text{(A 7)} \quad (\hat{a}_T + \hat{\rho}\hat{a}_{\rho T})\frac{\partial r}{\partial t} + \hat{\rho}\hat{a}_{T\rho s}\frac{\partial r_s}{\partial t} + \hat{\rho}\hat{a}_T\frac{\partial u_n}{\partial x} + \frac{\partial}{\partial y}(\hat{\rho}\hat{a}_T v_n) + \tfrac{1}{2}\alpha_2\left(\frac{\partial u_n}{\partial y}\right)^2 + \kappa\frac{\partial^2 \tau}{\partial y^2} = 0, \tag{14}$$

where the subscripts on $\hat{a}$ denote partial derivatives and the prime is the total derivative with respect to y. At the bottom of the liquid film we have the boundary conditions

$$u_n = v_n = v_s = 0, \quad \frac{\partial \tau}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{15}$$

where the generally accepted condition of no heat transfer across the solid/liquid interface is introduced.

The corresponding leading-order problem for the vapour is described by

$$\text{(A 10)} \quad \frac{1}{\Delta}\frac{\partial r_g}{\partial t} + \frac{\partial}{\partial x}(\hat{\rho}_g u_g) + \frac{\partial}{\partial y}(\hat{\rho}_g v_g) = 0, \tag{16}$$

$$\text{(A 11)} \quad \frac{\partial^2 u_g}{\partial y^2} = 0, \tag{17}$$

$$\text{(A 12)}\quad \frac{\partial p_{\mathrm{g}}}{\partial y}+\frac{r_{\mathrm{g}}}{\Delta^2}\omega' = 0, \tag{18}$$

$$\text{(A 13)}\quad p_{\mathrm{g}} = \hat{\rho}_{\mathrm{g}}(2\hat{a}_{\mathrm{g}\rho}+\hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{g}\rho\rho})\,r_{\mathrm{g}}, \tag{19}$$

$$\text{(A 14)}\quad \frac{1}{\Delta}(\hat{a}_{\mathrm{g}T}+\hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{g}T\rho})\frac{\partial r_{\mathrm{g}}}{\partial t}+\frac{\partial}{\partial x}(\hat{\rho}_{\mathrm{g}}\hat{a}_{\mathrm{g}T}u_{\mathrm{g}})+\frac{\partial}{\partial y}(\hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{g}T}\,v_{\mathrm{g}})+\kappa_{\mathrm{g}}\frac{\partial^2\tau_{\mathrm{g}}}{\partial y^2} = 0. \tag{20}$$

At the free surface (that is, on $y = 1$, to leading order) the vapour and liquid must satisfy the jump conditions:

$$\text{(A 15)}\quad \hat{\rho}\left(v_{\mathrm{n}}-\frac{\partial\eta}{\partial t}\right) = \sigma\Delta\hat{\rho}_{\mathrm{g}}\left(v_{\mathrm{g}}-\frac{1}{\Delta}\frac{\partial\eta}{\partial t}\right)\quad\left(=\frac{j}{\epsilon^2}\right), \tag{21}$$

$$\text{(A 16)}\quad p+\eta\hat{p}'-\sigma\Delta^2(p_{\mathrm{g}}+\eta\hat{p}'_{\mathrm{g}})+\beta\hat{\rho}'_{\mathrm{s}}\left[2\hat{\rho}_{\mathrm{s}}\left(\hat{\rho}''_{\mathrm{s}}\,\eta-\frac{\partial r_{\mathrm{s}}}{\partial y}\right)-\hat{\rho}'_{\mathrm{s}}(r_{\mathrm{s}}+\hat{\rho}'_{\mathrm{s}}\,\eta)\right] = 0, \tag{22}$$

$$\text{(A 17)}\quad \alpha_2\frac{\partial u_{\mathrm{n}}}{\partial y} = \sigma\Delta^2\alpha_{\mathrm{g}2}\frac{\partial u_{\mathrm{g}}}{\partial y}, \tag{23}$$

$$\text{(A 18)}\quad \frac{j}{\epsilon^2}(\Delta^2\hat{a}_{\mathrm{g}T}-\hat{a}_T) = \kappa\frac{\partial\tau}{\partial y}-\sigma\Delta^2\kappa_{\mathrm{g}}\frac{\partial\tau_{\mathrm{g}}}{\partial y}, \tag{24}$$

$$\text{(A 19)}\quad \frac{p+\eta\hat{p}'}{\hat{\rho}}-\frac{\Delta^2(p_{\mathrm{g}}+\eta\hat{p}'_{\mathrm{g}})}{\hat{\rho}_{\mathrm{g}}}+\left(\hat{a}_\rho-\frac{\hat{p}}{\hat{\rho}^2}\right)r+\left[\hat{\rho}'\left(\hat{a}_\rho-\frac{\hat{p}}{\hat{\rho}^2}\right)+\hat{\rho}'_{\mathrm{s}}\,\hat{a}_{\rho\mathrm{s}}\right]\eta+\hat{a}_{\rho\mathrm{s}}\,r_{\mathrm{s}}$$
$$+\frac{\tfrac{1}{2}\beta\hat{\rho}'_{\mathrm{s}}}{(\hat{\rho}\hat{\rho}_{\mathrm{s}})^2}\left[2\hat{\rho}\hat{\rho}_{\mathrm{s}}\left(\frac{\partial r_{\mathrm{s}}}{\partial y}+\hat{\rho}''_{\mathrm{s}}\,\eta\right)-\hat{\rho}_{\mathrm{s}}\hat{\rho}'_{\mathrm{s}}(r+\hat{\rho}'\eta)-\hat{\rho}\hat{\rho}'_{\mathrm{s}}(r_{\mathrm{s}}+\hat{\rho}'_{\mathrm{s}}\,\eta)\right] = 0, \tag{25}$$

$$\text{(A 21)}\quad u_{\mathrm{n}} = \Delta u_{\mathrm{g}},\quad \tau = \tau_{\mathrm{g}}, \tag{26}$$

where the equilibrium state ($3b$) for the vapour has been used in (25). By virtue of (21) and (24) it is clear that $j = O(\epsilon^2)$, and from (A 20) we see also that $\psi = O(\epsilon^2)$, which is then used in (A 19) to yield (25).

The analysis of the above equations is, for the most part, quite straightforward, and so we shall mention only some of the points that arise in their solution. The velocity components $u_{\mathrm{n}0}$ and $u_{\mathrm{g}0}$ turn out to take a very simple form, but only when the matching of $u_{\mathrm{g}0}$ as $y \to \infty$ is invoked. We shall discuss the matching in a little more detail later, but to continue our description of the linear wave we shall make use of the appropriate solution at this stage. The only available solutions for $u_{\mathrm{g}0}$ are either that $u_{\mathrm{g}0}$ is proportional to y as $y \to \infty$, or that $u_{\mathrm{g}0}$ is identically zero. The unbounded form is unmatchable, and so we obtain

$$u_{\mathrm{n}0} = u_{\mathrm{g}0} \equiv 0. \tag{27}$$

This choice accords with earlier theories (see e.g. Bergman 1969) where $u_{\mathrm{n}0} \equiv 0$ was usually an ingredient ('only the superfluid moves'), and then the vapour must satisfy $u_{\mathrm{g}0} \equiv 0$ since it is 'locked' to the normal fluid by the viscous stresses at the surface. The picture we are suggesting therefore involves the movement of the vapour – essentially normal – towards or away from the surface, this motion being driven by the condensation/evaporation. However, at large distances ($y \to \infty$) the relative magnitudes of v_{g} and u_{g} must interchange, allowing the recirculation of the vapour necessary to maintain mass conservation.

The pressures in the liquid and the vapour, at $y = 1$, are determined from (22) and (25) and yield, for example,

$$\varDelta^2 p_{g0} = \left\{\hat{\rho}_g \omega' \eta_0 + \frac{\frac{1}{2}\beta\hat{\rho}_g}{\hat{\rho} - \sigma\hat{\rho}_g}\left[\frac{\hat{\rho}_s'}{\hat{\rho}_s}\left(\hat{\rho}_s' r_{s0} - 2\hat{\rho}_s \frac{\partial r_{s0}}{\partial y}\right) - \left(\frac{\hat{\rho}_s'^2}{\hat{\rho}_s}\right)' \eta_0\right]\right\}_{y=1}, \tag{28}$$

with a similar expression for p_0 on $y = 1$. It is evident from (28) that we require the form of r_{s0}, and once this is known all the other variables follow directly. The equation for $r_{s0}(x, t, y)$ is obtained by eliminating r_0 between (11) and (13), and if we write $r_{s0} = \hat{\rho}_s'(\partial\theta_{s0}/\partial t)\, F(y)$, this gives

$$\beta \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\hat{\rho}_s'^2}{\hat{\rho}_s}\frac{\mathrm{d}F}{\mathrm{d}y}\right] = \hat{\alpha}\hat{\rho}_s'(\hat{a}_{\rho s} + \hat{\rho}\hat{a}_{\rho\rho s})\,(\omega' F - 1), \tag{29}$$

where $\hat{\alpha} = (2\hat{a}_\rho + \hat{\rho}\hat{a}_{\rho\rho})^{-1}$ is a measure of the compressibility of the liquid. (The incompressible limit corresponds to $\hat{\alpha} \to 0$.) The boundary conditions necessary for the complete determination of $F(y)$ require a little consideration. The equilibrium superfluid density $\hat{\rho}_s$ behaves like Y^2 as $Y \to 0^+$, where $Y \equiv y$ or $1 - y$ (see (2*d*) and (5)), and also the total ρ_s is to be zero on $y = 0,\ 1 + \epsilon\eta$. This would suggest that $F(y)$ merely be bounded at $y = 0, 1$, but then

$$\rho_s = \hat{\rho}_s + \epsilon r_{s0} + o(\epsilon) \quad \text{as} \quad \epsilon \to 0 \tag{30}$$

would not be uniformly valid as $y \to 0, 1$. However, an examination of the full equations leads us to the conclusion that there are no boundary layers for $y = O(\epsilon)$, $1 - y = O(\epsilon)$. Consequently (30) must be uniformly valid, and so $F(y)$ must (at most) approach zero linearly as $y \to 0,\ 1$; (29) permits such a solution, and in the case of a nearly incompressible liquid ($\hat{\alpha} \to 0$) an approximate solution is easily constructed which exhibits this behaviour.

It is now clear that p_0, r_0, r_{s0} are each proportional to $\partial\theta_{s0}/\partial t$, and then the four equations (10)–(13) cannot be independent, for consistency. That all is well is confirmed by making use of the equilibrium state; this property is no more than a repetition of that enjoyed by the equilibrium equations themselves. For convenience we set

$$r_{s0} = \frac{\partial\theta_{s0}}{\partial t} S_0(y), \quad r_0 = \frac{\partial\theta_{s0}}{\partial t} R_0(y), \tag{31}$$

where $R_0(y)$ is given by (11), whence

$$R_0 = -\hat{\alpha}[(\hat{a}_{\rho s} + \hat{\rho}\hat{a}_{\rho\rho s})\, S_0 + 1], \tag{32}$$

with $S_0 = \hat{\rho}_s' F$. The pressure p_0 is obtained by integrating (10) and using the boundary condition on $y = 1$ (see (28), following). This perturbation pressure will now contribute to the pressure on $y = 0$, and will in actuality alter the position of the solid/liquid interface. Nevertheless there are no grounds for believing that the purely local and small ($O(\epsilon)$) variation in substrate thickness has any significant effect upon the passage or form of the wave. Hereinafter we shall assume the model in which the liquid is bounded by $y = 0$ and $y = 1 + \epsilon\eta$, for all time, where $y = 0$ is treated as a solid fixed surface.

The relation between θ_{s0} and η_0 is easily obtained, for example, by evaluating (12) on $y = 1$, where it then takes the form

$$\frac{\partial\theta_{s0}}{\partial t} = \lambda_0 \eta_0, \tag{33}$$

where λ_0 is an involved constant depending on S_0 – and derivatives of S_0 – evaluated at $y = 1$. There is no virtue in presenting the general form of λ_0 here (see Appendix B); however, in the limit of an incompressible liquid, we obtain

$$\lambda_0 = \left[\tfrac{1}{2}\sigma\beta \frac{\hat{\rho}_{\mathrm{g}}\,(\hat{\rho}_{\mathrm{s}}'^2/\hat{\rho}_{\mathrm{s}})'}{\hat{\rho}(\hat{\rho}-\sigma\hat{\rho}_{\mathrm{g}})} - \omega'\right]_{y=1} \tag{34}$$

This result demonstrates how both healing ($\beta \neq 0$) and a vapour model ($\sigma \neq 0$) play a role even in the absence of liquid compressibility. (It must be emphasized that, purely from numerical estimates, the compressibility effects are rather more significant than this new phenomenon since σ is very small for helium II.)

The final stage in obtaining the equation of motion for the surface wave involves determining the contributions to the interfacial conditions. For the liquid this first requires the integration of (8) and (14) across the film to yield

$$[\hat{\rho}v_{\mathrm{n}}]_{y=1} + \int_0^1 \left(R_0 \frac{\partial^2\theta_{\mathrm{s}}}{\partial t^2} + \hat{\rho}_{\mathrm{s}} \frac{\partial^2\theta_{\mathrm{s}}}{\partial x^2}\right) \mathrm{d}y = 0 \tag{35}$$

and

$$\left[\kappa \frac{\partial\tau}{\partial y} + \hat{\rho}\hat{a}_T v_{\mathrm{n}}\right]_{y=1} + \frac{\partial^2\theta_{\mathrm{s}}}{\partial t^2} \int_0^1 (\hat{\rho}\hat{a}_{T\rho_{\mathrm{s}}} S_0 + (\hat{a}_T + \hat{\rho}\hat{a}_{\rho T})\, R_0]\, \mathrm{d}y = 0, \tag{36}$$

using $u_{\mathrm{s}0} = \partial\theta_{\mathrm{s}0}/\partial x$ (and $u_{\mathrm{n}0} = 0$). It is evident that this leading-order theory does not produce, nor does it require, the dependence of $v_{\mathrm{n}0}$ and τ_0 across the layer. These two functions are, nevertheless, completely determined at the next order, as the nonlinear theory will demonstrate. Correspondingly for the vapour, the relevant equations are to be integrated to give evaluations on $y = 1$. Although there is much to be said for attempting the integration from infinity, this turns out to be impossible. There is no automatic guarantee that the asymptotic expansions (of which (16)–(20) describe the first terms) are uniformly valid as $y \to 1$ (from infinity); furthermore, it is far from clear from a cursory examination that the leading order is even bounded as $y \to 1$. With these points in mind, we write

$$\hat{\rho}_{\mathrm{g}} v_{\mathrm{g}} = m_{\mathrm{g}} - \frac{1}{\Delta} \int_1^y \frac{\partial r_{\mathrm{g}}}{\partial t} \mathrm{d}y \tag{37}$$

and

$$\kappa_{\mathrm{g}} \frac{\partial\tau_{\mathrm{g}}}{\partial y} + \hat{\rho}_{\mathrm{g}} \hat{a}_{\mathrm{g}T} v_{\mathrm{g}} = q_{\mathrm{g}} + m_{\mathrm{g}} \hat{a}_{\mathrm{g}T1} - \frac{1}{\Delta} \int_1^y (\hat{a}_{\mathrm{g}T} + \hat{\rho}_{\mathrm{g}} \hat{a}_{\mathrm{g}\rho T}) \frac{\partial r_{\mathrm{g}}}{\partial t} \mathrm{d}y, \tag{38}$$

from (16) and (20) respectively, where $\hat{\rho}_{\mathrm{g}} v_{\mathrm{g}0} = m_{\mathrm{g}}$, $\kappa_{\mathrm{g}}\, \partial\tau_{\mathrm{g}0}/\partial y = q_{\mathrm{g}}$ both on $y = 1$, and $\hat{a}_{\mathrm{g}T1}$ denotes $\hat{a}_{\mathrm{g}T}$ evaluated on $y = 1$. For the purposes of obtaining the jump conditions across $y = 1$ we now require the (so-far unknown) functions m_{g} and q_{g}. However, m_{g} can be found in terms of η_0 without recourse to the behaviour of the vapour at infinity; from (21) and (35) it follows that

$$\sigma\Delta m_{\mathrm{g}} = [\sigma\hat{\rho}_{\mathrm{g}} - \hat{\rho}]_{y=1} \frac{\partial\eta}{\partial t} - \int_0^1 \left(R_0 \frac{\partial^2\theta_{\mathrm{s}}}{\partial t^2} + \hat{\rho}_{\mathrm{s}} \frac{\partial^2\theta_{\mathrm{s}}}{\partial x^2}\right) \mathrm{d}y. \tag{39}$$

This can be interpreted as specifying the form of m_{g} necessary to maintain the wave motion: the wave elevation/depression has an associated cooling/heating effect which drives the condensation/evaporation, which can only be maintained by an appropriate vapour flow to/from the surface.

The equation for $\eta_0(x,t)$ is now obtained by eliminating j/ϵ^2 between (21) and (24), and making use of (33), (35) and (36), to yield

$$\left[\frac{\hat{\rho}_1}{\lambda_0}(\Delta^2\hat{a}_{\mathrm{g}T1}-\hat{a}_{T1})-\int_0^1\{\hat{\rho}\hat{a}_{T\rho\mathrm{s}}S_0+(\hat{a}_T+\hat{\rho}\hat{a}_{T\rho}-\Delta^2\hat{a}_{\mathrm{g}T1})R_0\}\,\mathrm{d}y\right]\frac{\partial^2\eta_0}{\partial t^2}$$
$$-(-\Delta^2 a_{\mathrm{g}T1})\left(\int_0^1\hat{\rho}_\mathrm{s}\,\mathrm{d}y\right)\frac{\partial^2\eta_0}{\partial x^2}-\frac{\sigma\Delta^3}{\lambda_0}\frac{\partial q_\mathrm{g}}{\partial t}=0, \quad (40)$$

where the additional subscript 1 again denotes evaluation on $y=1$. Equation (40) cannot be further determined without constructing a representation for q_g, the heat transfer at the surface arising from the flow of the vapour. Thus (40) constitutes the most general (linear) equation of motion for η_0 that can be obtained from our theory: our further comments will specialize (40) somewhat.

Some discussion of the vapour is now necessary, in particular as to its structure far from the surface where an appropriate 'outer' expansion is valid. To find where this outer structure exists we can note one or two properties of the solution of (16)–(20) as $y\to\infty$. First, it is clear from (18) and (19) that both $p_{\mathrm{g}0}$ and $r_{\mathrm{g}0}$ approach an ambient state (exponentially) as $y\to\infty$, and that both are proportional to η_0/Δ^2 (see (28)). On the other hand we have that $v_{\mathrm{g}0}$ is proportional to y, and $\tau_{\mathrm{g}0}$ to y^2, as $y\to\infty$, and this algebraic behaviour leads to a breakdown when $y=O(\epsilon^{-\frac{1}{2}})$. (It is also confirmed from the governing equations that this scale is consistent in the balancing of appropriate terms.) Although we have argued in the absence of the terms in $u_{\mathrm{g}0}$, their retention produces exactly the same breakdown but now with $u_\mathrm{g}=O(\epsilon^{-\frac{1}{2}})$, which does not generate a consistent balance in the full equations. The first term for u_g is therefore $\epsilon u_{\mathrm{g}1}$, which is $O(1)$ in the outer region, and so the outer variables are defined by

$$y=\epsilon^{-\frac{1}{2}}Y,\quad u_\mathrm{g}=U_\mathrm{g},\quad v_\mathrm{g}=\epsilon^{-\frac{1}{2}}V_\mathrm{g}, \quad (41)$$
$$p_\mathrm{g}=\hat{p}_\mathrm{g}+\epsilon P_\mathrm{g},\quad \rho_\mathrm{g}=\hat{\rho}_\mathrm{g}+\epsilon R_\mathrm{g},\quad \tau_\mathrm{g}=1+\epsilon T_\mathrm{g},$$

where the equilibrium state $(\hat{p}_\mathrm{g},\hat{\rho}_\mathrm{g})$ is a constant with exponentially small correction terms (as $\epsilon\to 0$); see (2). It is not our aim here to give a completely detailed picture of the vapour: this would require being specific, for example, on the question of boundary conditions (if any) to be imposed for $Y=O(1)$. To complete the description of (40) we merely require the form of q_g, in particular its dependence on η_0 (and the parameters σ, Δ). The equations valid in the outer region of the vapour, correct to $O(\epsilon)$, are

$$\frac{1}{\Delta}\frac{\partial R_\mathrm{g}}{\partial t}+\hat{\rho}_\mathrm{g}\left(\frac{\partial U_\mathrm{g}}{\partial x}+\frac{\partial V_\mathrm{g}}{\partial Y}\right)=0,\quad \hat{\rho}_\mathrm{g}\frac{\partial U_\mathrm{g}}{\partial t}=-\frac{\partial P_\mathrm{g}}{\partial x}+\tfrac{1}{2}\alpha_{2\mathrm{g}}\frac{\partial^2 U_\mathrm{g}}{\partial Y^2},\quad \frac{\partial P_\mathrm{g}}{\partial Y}=0, \quad (42a,b,c)$$

$$P_\mathrm{g}=\hat{\rho}_\mathrm{g}(2\hat{a}_{\mathrm{g}\rho}+\hat{\rho}\hat{a}_{\mathrm{g}\rho\rho})R_\mathrm{g}+\hat{\rho}_\mathrm{g}^2\hat{a}_{\mathrm{g}\rho T}T_\mathrm{g}, \quad (42d)$$

$$\frac{1}{\Delta}\frac{\partial}{\partial t}[(\hat{a}_{\mathrm{g}T}+\hat{\rho}_\mathrm{g}\hat{a}_{\mathrm{g}\rho T})R_\mathrm{g}+\hat{\rho}_\mathrm{g}\hat{a}_{\mathrm{g}TT}T_\mathrm{g}]+\hat{\rho}_\mathrm{g}\hat{a}_{\mathrm{g}T}\left(\frac{\partial U_\mathrm{g}}{\partial x}+\frac{\partial V_\mathrm{g}}{\partial Y}\right)+\kappa_\mathrm{g}\frac{\partial^2 T_\mathrm{g}}{\partial Y^2}=0. \quad (42e)$$

(Note that here, as before, we have omitted zero subscripts on the dependent variables.)

It is immediately clear that, to match, P_g must take on the ambient form of $p_{\mathrm{g}0}$ (which is independent of y), and thus P_g contains a contribution $O(1)$, but not $O(\epsilon^{\frac{1}{2}})$ as well, even though valid to $O(\epsilon)$. We should remember that our choice is to construct the basic expansion using the asymptotic sequence $\{\epsilon^n\}$ (see (6) and §2). Thus from (42e), after introducing (42a, d), we obtain

$$T_{\mathrm{g}} = T_{\mathrm{g}0}(x,t,Y)+O(\epsilon),$$

which permits matching with $\kappa_{\mathrm{g}}\,\partial\tau_{\mathrm{g}}/\partial y$ (see (38)) only if the term independent of y is set to zero, a necessity by virtue of the absence of a term $O(\epsilon^{\frac{1}{2}})$ in the expansion of τ_{g}. Hence from (38) we are able to determine q_{g} as

$$q_{\mathrm{g}} = \frac{1}{\sigma\mathit{\Delta}}(\hat{a}_{\mathrm{g}T1}-\hat{a}_{\mathrm{g}T\infty})\left[(\hat{\rho}_1-\sigma\hat{\rho}_{\mathrm{g}1})\frac{\partial\eta_0}{\partial t}+\int_0^1\left(R_0\frac{\partial^2\theta_{\mathrm{s}0}}{\partial t^2}+\hat{\rho}_{\mathrm{s}}\frac{\partial^2\theta_{\mathrm{s}0}}{\partial x^2}\right)\mathrm{d}y\right]$$
$$+\frac{1}{\mathit{\Delta}}\frac{\partial}{\partial t}\int_1^\infty[(\hat{a}_{\mathrm{g}T}+\hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{g}\rho T}-\hat{a}_{\mathrm{g}T\infty})\,r_{\mathrm{g}0}-\hat{\rho}_{\mathrm{g}\infty}\,\hat{a}_{\mathrm{g}\rho T\infty}\,r_{\mathrm{g}0\infty}]\,\mathrm{d}y-\frac{1}{\mathit{\Delta}}\hat{\rho}_{\mathrm{g}\infty}\,\hat{a}_{\mathrm{g}\rho T\infty}\frac{\partial r_{\mathrm{g}0\infty}}{\partial t}, \quad (43)$$

where the linearly increasing term in $\partial\tau_{\mathrm{g}0}/\partial y$ has been removed, and otherwise the integrals in (43) are assumed finite; the subscript ∞ denotes evaluation at the ambient state, $y\rightarrow\infty$. It is convenient to rewrite q_{g} so as to yield

$$\frac{\partial q_{\mathrm{g}}}{\partial t} = \frac{1}{\sigma\mathit{\Delta}}(\hat{a}_{\mathrm{g}T1}-\hat{a}_{\mathrm{g}T\infty})\left[\left(\hat{\rho}_1-\sigma\hat{\rho}_{\mathrm{g}1}+\lambda_0\int_0^1 R_0\,\mathrm{d}y\right)\frac{\partial^2\eta_0}{\partial t^2}+\lambda_0\left(\int_0^1\hat{\rho}_{\mathrm{s}}\,\mathrm{d}y\right)\frac{\partial^2\eta_0}{\partial x^2}\right]+\frac{n_0}{\mathit{\Delta}^3}\frac{\partial^2\eta_0}{\partial t^2}, \quad (44)$$

where use has been made of (33) and the fact that $r_{\mathrm{g}0}\propto\mathit{\Delta}^{-2}\eta_0$; (43) essentially defines n_0 (it is the contribution to q_{g} from the flow of entropy in the vapour). Before the equation for $\eta_0(x,t)$ is written down in its final form, it should be emphasized that alternatives to (43) could be found. It we introduce specific boundary conditions at $Y = O(1)$ (or appropriate matching conditions if the boundary is at $Y^{-1} = o(1)$), which allow a term $O(\epsilon^{\frac{1}{2}})$ in the outer expansion, then we could arrange that $q_{\mathrm{g}} = 0$, for example. Of course, such a result would be exceptional and therefore not likely to be relevant to the experimental configuration. Nevertheless, within the limitations of the asymptotic expansions assumed here, we have produced (43) as the most reasonable choice: certainly any other possibility requires the boundary/matching conditions to *force* a term $O(\epsilon^{\frac{1}{2}})$ into the expansions.

The linear wave equation for $\eta_0(x,t)$ is obtained by eliminating $\partial q_{\mathrm{g}}/\partial t$ between (44) and (40) to yield

$$\Bigg[\frac{\hat{\rho}_1}{\lambda_0}(\mathit{\Delta}^2\hat{a}_{\mathrm{g}T\infty}-\hat{a}_{T1})+\frac{\sigma\hat{\rho}_{\mathrm{g}1}}{\lambda_0}\mathit{\Delta}^2(\hat{a}_{\mathrm{g}T1}-\hat{a}_{\mathrm{g}T\infty})-\frac{\sigma n_0}{\lambda_0}$$
$$-\int_0^1\{\hat{\rho}\hat{a}_{T\rho\mathrm{s}}S_0+(\hat{a}_T+\hat{\rho}\hat{a}_{T\rho}-\mathit{\Delta}^2\hat{a}_{\mathrm{g}T\infty})\,R_0\}\,\mathrm{d}y\Bigg]\frac{\partial^2\eta_0}{\partial t^2}$$
$$-(-\mathit{\Delta}^2\hat{a}_{\mathrm{g}T\infty})\left(\int_0^1\hat{\rho}_{\mathrm{s}}\,\mathrm{d}y\right)\frac{\partial^2\eta_0}{\partial x^2} = 0. \quad (45)$$

(The complete details of this equation can be found in Appendix B.) It is convenient to introduce the 'latent heats'

$$l_1 = \hat{a}_{T1}-\mathit{\Delta}^2\hat{a}_{\mathrm{g}T1}, \quad l_\infty = \hat{a}_{T1}-\mathit{\Delta}^2\hat{a}_{\mathrm{g}T\infty}, \quad (46)$$

and then a useful reduced version of (45) for small σ (with l_1, l_∞ fixed) gives a wave speed c, where

$$c^2 \approx \frac{\omega_1'(l_\infty-\hat{a}_{T1})\displaystyle\int_0^1\hat{\rho}_{\mathrm{s}}\,\mathrm{d}y}{l_\infty[\hat{\rho}+\beta(\hat{\rho}_{\mathrm{s}}''S_0-\hat{\rho}_{\mathrm{s}}'S_0')/\hat{\rho}_{\mathrm{s}}]_1-\omega_1'\displaystyle\int_0^1 A_0\,\mathrm{d}y}, \quad (47)$$

with $A_0 = \hat{\rho}\hat{a}_{T\rho s}S_0 + (\hat{a}_T + \hat{\rho}\hat{a}_{T\rho} - \Delta^2\hat{a}_{gT\infty})R_0$. The wave speeds given by (47), and more completely as implied by (45), are generalizations of those given by Atkins & Rudnick (1970), Putterman (1974) and Johnson (1978*a*, 1984). The speed obtained from (47) is probably the most useful in practice since the value of σ for helium II is about 10^{-3}, and therefore the terms associated with the vapour density presumably may be ignored. The further restriction to nearly incompressible flow in the liquid involves S_0, $R_0 \rightarrow 0$ (they are both proportional to $\hat{\alpha}$; see (29) and (32)), whence

$$c^2 \approx \frac{\omega_1'}{\hat{\rho}_1 l_1}(l_\infty - \hat{a}_{T1}) \int_0^1 \hat{\rho}_s \, \mathrm{d}y,$$

essentially the 'classical' result (see Putterman 1974). Although it is certainly possible – and also perhaps reasonable – to offer simplified formulae (for c) on the basis of numerical estimates, that is not the thrust of the argument presented here. Equation (45) (and see Appendix B) gives a quite general wave speed which incorporates the effects of compressibility, healing and vapour flow; note that any consequences of the relaxation phenomenon are absent in the linear long-wave theory.

Further details on the form that the vapour takes in the outer region can be obtained from (42). In particular the recirculation is evident if these equations are solved with a boundary condition $V_g = 0$ on $Y = Y_0$ (or matching condition $V_g \rightarrow 0$ as $Y \rightarrow \infty$). Nevertheless, as we have demonstrated, the complete resolution of the vapour problem is unnecessary if only the leading-order contribution q_g to the surface wave is required. Consequently we shall not pursue this analysis any further in the current presentation.

4. A nonlinear theory

The nonlinear theory under consideration here comes about as the natural extension to the linear theory. With the choice $\delta = \epsilon$ already mentioned, and linear propagation at a speed c, where

$$\frac{\partial^2 \eta_0}{\partial t^2} - c^2 \frac{\partial^2 \eta_0}{\partial x^2} = 0$$

(see (45)), a far-field theory can be constructed in the usual manner by introducing

$$\xi = x - ct, \quad \mathscr{T} = \epsilon t. \tag{48}$$

The limit process $\epsilon \rightarrow 0$ will then generate a corresponding nonlinear equation describing $\eta_0(\xi, \mathscr{T})$ which will presumably incorporate a variety of contributions associated with the parameters that are held fixed. Because of the chosen form of limiting procedure, we are able to argue that the resulting equation is in some sense the most general one applicable to small-amplitude long waves on helium II. Of course, many other equations are available – under other limiting procedures – but only at the expense of choosing more parameters to be appropriate functions of ϵ, an approach which is perhaps questionable on physical grounds.

The complexities of the analysis suggested above are immediately evident: not only are the governing equations (to $O(\epsilon^2)$) very involved, but in particular many additional nonlinear terms are produced by expanding about $y = 1$ those functions defined on $y = 1 + \epsilon\eta$. There is very little virtue in obtaining explicitly all the coefficients of the equation for η_0, and in fact much of the work would eventually involve extensive computations. We shall therefore restrict ourselves to an outline of the form of the equation but indicate specifically how the viscous and relaxation effects are represented in it. The details of the many contributions to the nonlinear terms will be altogether suppressed although the existence of these terms will be noted.

In the region where $y = O(1)$, it is assumed that all the dependent variables may be expressed as asymptotic expansions of the form

$$Q = Q_0 + \epsilon Q_1 + o(\epsilon)$$

as $\epsilon \to 0$. The leading order produces an identity since we are making use of the linear wave speed c but the next order then enables the equation for $\eta_0(\xi, \mathscr{T})$ to be determined. The method of solution follows the outline given in §3, but now with the bonus that $v_{\mathrm{n}0}$, τ_0 and $v_{\mathrm{s}0}$ are also completely determined. Thus, for example, we can find p_1 and $p_{\mathrm{g}1}$ at the interface, and if we eliminate p_1, r_1 and $r_{\mathrm{s}1}$ between the $O(\epsilon)$ equations obtained from (A 3)–(A 6) the resulting consistency condition defines $v_{\mathrm{n}0}$. This corresponds to the interdependence of the four equations (10)–(13). The equation for $v_{\mathrm{n}0}$ is

$$\kappa\left[(\alpha_1+\alpha_2)\frac{\partial^2 v_{\mathrm{n}0}}{\partial y^2} + \gamma\hat{\rho}_{\mathrm{n}}\frac{\partial}{\partial y}\left\{\hat{\rho}_{\mathrm{s}}^{-\frac{1}{2}}\frac{\partial}{\partial y}(\hat{\rho}_{\mathrm{n}} v_{\mathrm{n}0})\right\}\right] - (\hat{\rho}\hat{a}_T)^2 v_{\mathrm{n}0}$$
$$+ c\hat{\rho}\hat{a}_T\frac{\partial}{\partial \xi}\int_0^y [r_0(\hat{a}_T + \hat{\rho}\hat{a}_{T\rho}) + r_{\mathrm{s}0}\hat{\rho}\hat{a}_{T\rho\mathrm{s}}]\,\mathrm{d}y - c\kappa\gamma\hat{\rho}_{\mathrm{n}}\frac{\partial^2}{\partial y\,\partial \xi}(r_0 - r_{\mathrm{s}0}) = 0, \quad (49)$$

with the boundary conditions

$$v_{\mathrm{n}0} = 0 \quad \text{on} \quad y = 0, \quad v_{\mathrm{n}0} = \frac{1}{\hat{\rho}}\frac{\partial^2 \theta_{\mathrm{s}0}}{\partial \xi^2}\int_0^1 (c^2 R_0 - \hat{\rho}_{\mathrm{s}})\,\mathrm{d}y \quad \text{on} \quad y = 1, \quad (50)$$

this latter condition being just (35). Now (36) can be used to obtain τ_0, and then $v_{\mathrm{s}0}$ is deduced from (8). Equation (49) is a generalization of that given by Johnson (1978*b*) for a related problem in heat transfer at a boundary. To make this comparison we consider the incompressible limit ($r_0, r_{\mathrm{s}0} \to 0$, $\hat{\rho} \to$ constant) in the absence of healing (whence all the coefficients are constants), yielding

$$\kappa\left[\alpha_1 + \alpha_2 + \frac{\gamma(\hat{\rho}-\hat{\rho}_{\mathrm{s}})^2}{\hat{\rho}_{\mathrm{s}}^{\frac{1}{2}}}\right]\frac{\partial^2 v_{\mathrm{n}0}}{\partial y^2} - (\hat{\rho}\hat{a}_T)^2 v_{\mathrm{n}0} = 0.$$

The lengthscale associated with the solution of this equation is

$$\left[\kappa(\alpha_1+\alpha_2) + \kappa\gamma\frac{(\hat{\rho}-\hat{\rho}_{\mathrm{s}})^2}{\hat{\rho}_{\mathrm{s}}^{\frac{1}{2}}}\right]^{\frac{1}{2}} \Big/ |\hat{\rho}\hat{a}_T|,$$

which is itself a generalization of the Clark thickness (see Roberts & Donnelly 1974; Johnson 1978*b*). In other words, the normal velocity component across the layer, to leading order, incorporates all the effects in our model: heat conduction, viscosity, relaxation and healing (because the coefficients depend on y). Other than using the simplifications indicated above, we can solve (49) only numerically given appropriate thermodynamic data.

As with the near-field (linear) theory, the analysis is fairly straightforward once $r_{\mathrm{s}1}$ has been determined, although the details of the outer region of the vapour and its matching could prove tiresome. (Since our concern can only be with the generalities surrounding the far-field theory, the problem of the vapour will be avoided by introducing certain simplifying assumptions to be mentioned later.) The equation for r_{s1} is obtained by eliminating r_1 between the appropriate forms deduced from (A 4) and (A 6), giving rise to the equation

$$\frac{\beta}{\hat{\rho}_s^2}\left[\hat{\rho}_s\frac{\partial^2 r_{s1}}{\partial y^2}-\hat{\rho}_s'\frac{\partial r_{s1}}{\partial y}+\frac{r_{s1}}{\hat{\rho}_s}(\hat{\rho}_s'^2-\hat{\rho}_s\hat{\rho}_s'')\right]+r_{s1}(\hat{\alpha}\hat{\alpha}_s^2-\hat{\rho}\hat{a}_{\rho s\,\rho s}$$

$$+\hat{\alpha}\hat{\alpha}_s\left(\frac{\partial\theta_{s0}}{\partial\mathscr{T}}-c\frac{\partial\theta_{s1}}{\partial\xi}\right)+\tau_0[\hat{\alpha}\hat{\alpha}_s(\hat{a}_T+\hat{\rho}\hat{a}_{T\rho})-\hat{\rho}\hat{a}_{T\rho s}]-\gamma m_s(1+\hat{\alpha}\hat{\alpha}_s)+C_1\,\eta_0^2=0, \quad (51)$$

where

$$\hat{\alpha}_s=\hat{a}_{\rho s}+\hat{\rho}\hat{a}_{\rho\rho s}$$

and

$$m_s=\left[-c\frac{\partial r_{s0}}{\partial\xi}+\hat{\rho}_s\frac{\partial u_{s0}}{\partial\xi}+\frac{\partial}{\partial y}(\hat{\rho}_s v_{s0})\right]\Big/\hat{\rho}_s^{\frac{1}{2}}.$$

The term $C_1(y)\,\eta_0^2$ represents the nonlinear contribution to the equation, and the equation is to be solved subject to $r_{s1}\to 0$ as $y\to 0$, $r_{s1}\to -\frac{1}{2}\eta_0^2\hat{\rho}_s''$ as $y\to 1$. From the solutions for v_{n0}, τ_0 and v_{s0} (which we suppose are available) we can observe that both τ_0 and v_{s0} are proportional to $\partial\eta_0/\partial\xi$, and thus we express the solution of (51) in the form

$$r_{s1}=-c\frac{\partial\theta_{s1}}{\partial\xi}S_{11}+\frac{\partial\theta_{s0}}{\partial\mathscr{T}}S_{12}+\frac{\partial\eta_0}{\partial\xi}(S_{13}+\gamma S_{14})+\eta_0^2 S_{15}, \quad (52)$$

where $S_{1i}=S_{1i}(y)$, and the term S_{14} arising from the relaxation process is isolated. Note that, although $\hat{\rho}_s\to 0$ as $y\to 0$, 1, m_s remains finite in these limits. Corresponding to (31), we have a similar solution for r_1:

$$r_1=-c\frac{\partial\theta_{s1}}{\partial\xi}R_{11}+\frac{\partial\theta_{s0}}{\partial\mathscr{T}}R_{12}+\frac{\partial\eta_0}{\partial\xi}(R_{13}+\gamma R_{14})+\eta_0^2 R_{15}, \quad (53)$$

where $R_{1i}=R_{1i}(y)$. The relationship between θ_{s1} and η_1 derives from p_1 evaluated on $y=1$ (cf. (33)) which may be expressed as

$$-c\frac{\partial\theta_{s1}}{\partial\xi}=\lambda_{11}\eta_1+\lambda_{12}\frac{\partial\theta_{s0}}{\partial\mathscr{T}}+(\lambda_{13}+\gamma\lambda_{14}+(\alpha_1+\alpha_2)\lambda_{15})\frac{\partial\eta_0}{\partial\xi}+\lambda_{16}\eta_0^2, \quad (54)$$

where the λ_{1i} are constants, and the terms associated with relaxation and viscous dissipation have been noted. At this stage it should be remembered that the linear theory implies the identities $S_{11}\equiv S_0$, $R_{11}\equiv R_0$ and $\lambda_{11}=\lambda_0$; further, the relation between θ_{s0} and η_0 in the far field is (see (33))

$$-c\frac{\partial\theta_{s0}}{\partial\xi}=\lambda_0\eta_0. \quad (55)$$

Finally, combining the relevant forms of the equations of mass and energy conservation, and introducing the free-surface boundary conditions, we obtain

$$\int_0^1\left[c\frac{\partial}{\partial\xi}\{\hat{\rho}(r_{s1}\hat{a}_{T\rho s}+r_1\hat{a}_{T\rho}+\tau_0\hat{a}_{TT})+r_1\hat{a}_T\}-\frac{\partial}{\partial\xi}(u_{n1}\hat{\rho}\hat{a}_T)\right.$$

$$\left.-\frac{\partial}{\partial\mathscr{T}}\{r_0(\hat{a}_T+\hat{\rho}\hat{a}_{T\rho})+r_{s0}\hat{\rho}\hat{a}_{T\rho s}\}\right]\mathrm{d}y-\sigma\varDelta^3 q_{g1}$$

$$=\hat{\rho}_1\left(c\frac{\partial\eta_1}{\partial\xi}-\frac{\partial\eta_0}{\partial\mathscr{T}}\right)(\varDelta^2\hat{a}_{gT1}-\hat{a}_{T1})+K_1\,\eta_0\,\eta_{0\xi}$$

$$+\varDelta^2\hat{a}_{gT1}\int_0^1\left[c\frac{\partial r_1}{\partial\xi}-\frac{\partial r_0}{\partial\mathscr{T}}-\frac{\partial}{\partial\xi}(\hat{\rho}_n u_{n1}+r_{s0}u_{s0}+\hat{\rho}_s u_{s1}]\right]\mathrm{d}y, \quad (56)$$

where K_1 is a constant. The heat transfer to/from the gas at the free surface is defined by

$$\left[\kappa_{\mathrm{g}}\frac{\partial\tau_{\mathrm{g1}}}{\partial y}+\hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{gT}\rho}\,r_{\mathrm{g0}}\,v_{\mathrm{g0}}\right]_{y=1}=q_{\mathrm{g1}},$$

which is to be compared with (38), following. To obtain the evolution equation for η_0, apart from substituting for the functions $r_0, r_{\mathrm{s0}}, \tau_0, r_1$ and r_{s1} (which we assume are known) into (56), requires some knowledge of u_{n1}, u_{s1} and q_{g1}. Of course $u_{\mathrm{s1}} = \partial\theta_{\mathrm{s1}}/\partial\xi$, and all the terms in θ_{s1} and η_1 will anyway cancel identically by virtue of the linear theory. The general form of u_{n1} is easily obtained as

$$u_{\mathrm{n1}}=\frac{1}{\alpha_2}U_1(y)\frac{\partial\eta_0}{\partial\xi}, \tag{57}$$

and we again assume that $U_1(y)$ is uniquely determined by the boundary conditions on $y = 0, 1$ (which involve u_{g1}, and so successful matching to the outer regions of the vapour is also implied). The explicit dependence on the viscosity is incorporated (see (57)) so that the consequences of allowing $\alpha_2 \to 0$ may be investigated. For simplicity we shall take $q_{\mathrm{g1}} \equiv 0$, and then (56) can certainly be written in the form

$$\frac{\partial\eta_0}{\partial\mathscr{T}}+K\eta_0\frac{\partial\eta_0}{\partial\xi}=\left(c_1+\gamma c_2+(\alpha_1+\alpha_2)\,c_2+\frac{c_4}{\alpha_2}\right)\frac{\partial^2\eta_0}{\partial\xi^2}, \tag{58}$$

where K and c_i are constants. Thus the small-amplitude long wave on helium II satisfies the Burgers equation in the far field, although expressions for the constants in (58) are not readily available.

We have attempted here mainly to outline the argument underlying the development of the nonlinear theory. If the analysis justifies further examination, and suitable thermodynamic data are available, then the detailed coefficients required for (58) can be obtained. The important aspects of the analysis are, first, the observation that in an appropriate neighbourhood of the wave front the Burgers equation is always relevant, and, secondly, that $\alpha_2 \to 0$ is a singular limit. This latter point is hardly a surprise since a crucial feature of third sound is the essentially stationary character of the normal fluid, i.e. the viscous forces predominate. This means, of course, that any theory of third sound which attempts to ignore the viscosity of the normal fluid in a nonlinear theory is open to some criticism. It is clear that no further useful information can be gleaned from this nonlinear theory without extensive and tedious analysis.

5. Some numerical results

The point was made in §1 that only a modest amount of numerical work can be realistically undertaken at the present time. The extremely involved nature of the thermodynamics of helium II means that it is not at all straightforward to represent it in a suitable numerical form. Thus, although predictions for the wave speed c are of considerable interest, even these can not yet be found with any reliability when all the effects discussed here are retained. We shall therefore present only a few results, but they will contain a measure of dependability. However, for completeness, some estimates for the wave speed will be mentioned. In the long term it is hoped that the theory discussed here will be tested by the use of accurate expressions for the Helmholtz free energy (and its derivatives). Certainly, if simplified models for $A(\rho, \rho_{\mathrm{s}}, T)$ are of interest, then the calculations should be fairly straightforward.

The Helmholtz free energy A – the fundamental thermodynamic quantity – is modelled as a polynomial of degree seven in both ρ and ρ_s, at fixed temperature. This was accomplished by applying 'best-fit' routines to the data of Brooks & Donnelly (1977), and that of Donnelly, Hills & Roberts (1978, private communication). The former paper is concerned only with helium II in its bulk state, so that A does not vary with ρ_s at fixed ρ and T. The latter work gives the variation of A with ρ_s, but only A, and also at rather large increments in temperature. Thus, although data for A_T and A_{TT} are directly available for the bulk liquid, the same cannot be said of the general liquid. The polynomials turned out to be sufficiently accurate† to allow differentiation with respect to ρ and ρ_s (and checks were performed against the more complete data for the bulk liquid). It is of course the restriction on the polynomials, being at fixed temperature, which ultimately limits what we can do numerically.

The polynomial description for A was used in (2) to find the equilibrium state of the liquid layer. This involved an iteration procedure through the equations together with an iteration on (2d) to determine $\hat{\rho}_s(y)$. The calculations were performed at fixed temperature and for a given film thickness h, using the appropriate pressure boundary conditions. The method also required the thickness of the solid phase to be determined as well as $\hat{\rho}(y)$ through the layer; the integration to find $\hat{\rho}_s(y)$ was accomplished by a Newton–Raphson–Kantarovich routine. This whole procedure, although rather involved, proved to converge rapidly and was insensitive to the initial guess for $\hat{\rho}_s(y)$, provided this was not too near zero. (The calculation was usually initiated by setting $\hat{\rho}_s(y)$ equal to the bulk value at the given temperature, even though the guess did not satisfy the boundary conditions on $y = 0, 1$; other guesses were also used to test the sensitivity.)

Now the integration for $\hat{\rho}_s$ gave rise to an interesting and exceptional situation which has implications in view of certain observations related to thin films. For sufficiently thin films the residual (which measures the rate of convergence) *increased* for the first few iteration loops, rather than decreasing significantly as happened for all other film thicknesses. The solution then moved towards $\hat{\rho}_s(y) \equiv 0$, eventually converging quite rapidly to this solution. It is clear that (2d) has the zero solution (and hence the reason why the initial guess for $\hat{\rho}_s(y)$ must not be too close to zero). This result suggests that, for h less than some critical value $h_c(T)$, a non-zero solution for $\hat{\rho}_s$ does not exist. When a critical value was suspected, great care was taken in the initial guess for $\hat{\rho}_s(y)$ – for example, by starting with the solution already obtained for a slightly larger value of h. This enabled the value of h_c to be isolated within fairly narrow bounds. Moreover the existence of only the zero solution for $h < h_c$, far from being alarming, is definitely a success for the continuum theory as we shall see. Typical results obtained from the numerical work are given in figures 2, 3 and 4, showing the superfluid density distribution, the total density and the superfluid fraction defined by

$$\int_0^1 \hat{\rho}_s \, dy \Big/ \int_0^1 \hat{\rho} \, dy$$

(when expressed in non-dimensional variables). All the calculations were undertaken for two temperatures: 1.3 and 1.8 K.

Although it is beyond our reasonable means to compute accurate predictions for the wave speed c, it behoves us to make some mention of what is – after all – intrinsic to third sound. The best we can offer is an estimate for c based on the calculations

---

† In terms of variables normalized with respect to maximum values the root-mean-square error was kept below 10^{-6}.

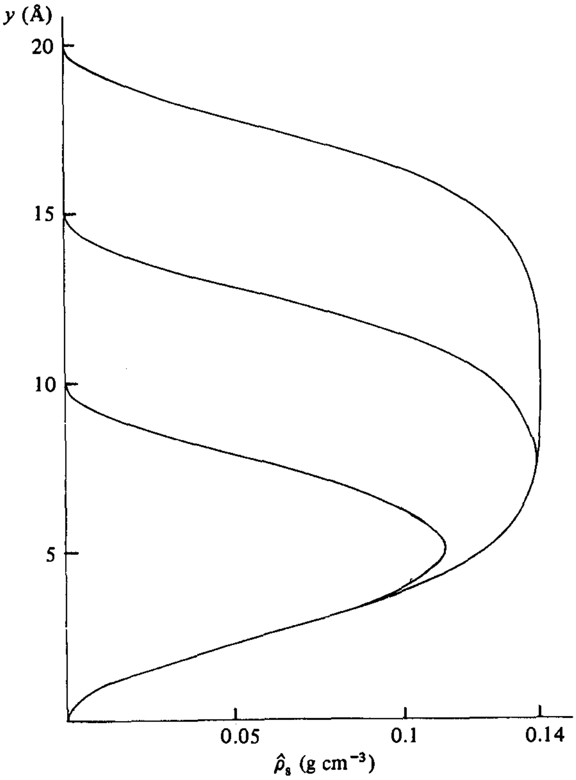

FIGURE 2. The equilibrium superfluid density at $T = 1.3$ K for three film thicknesses: 10, 15, 20 Å.

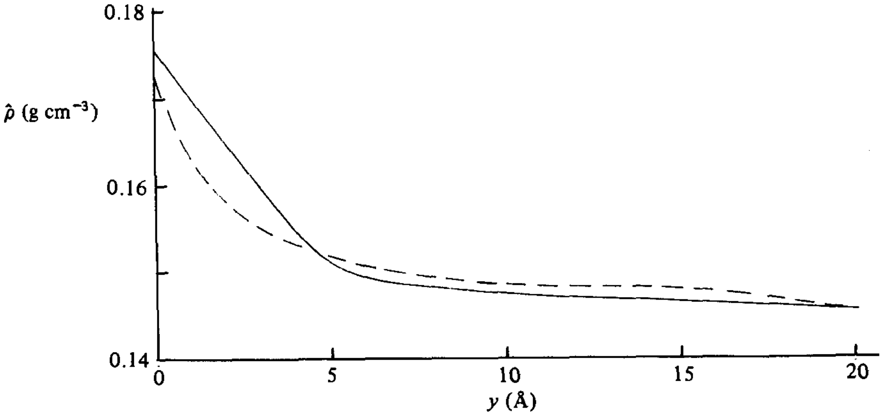

FIGURE 3. The total density as a function of depth for a film thickness of 20 Å at two temperatures: $T = 1.8$ K (——), $T = 1.3$ K (——).

of Johnson (1978*a*) (these incorporate compressibility but not healing) and then to introduce our current values for $\int_0^1 \hat{\rho}_s \, \mathrm{d}y$, which do include healing. In other words the terms in c (as implied by (45): see Appendix B) that arise from both healing *and* compressibility are ignored (i.e. $S_0 \equiv 0$), and we shall also assume that σ is negligibly

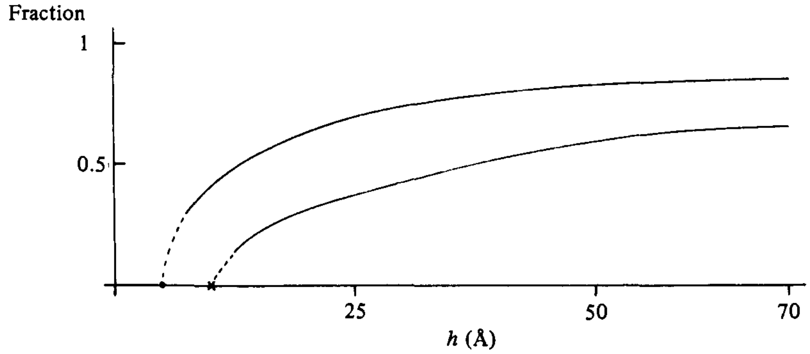

FIGURE 4. The superfluid fraction as a function of film thickness at $T = 1.3$ K (upper curve), $T = 1.8$ K (lower curve). The superfluid fraction is zero at ● ($T = 1.3$ K) and × ($T = 1.8$ K).

small. These wave speeds are shown in figures 5 and 6, together with the speeds without healing and the experimental data deduced from Putterman (1974) at roughly comparable temperatures.

6. Discussion

It has been demonstrated that, although in places quite involved, a linear theory for third sound can be developed under minimal assumptions. The waves of interest – small amplitude, long wavelength – have been isolated without the necessity of imposing any limiting process on the other parameters. In this sense we have presented a general theory, and a similar comment applies to the coupling between the liquid and the vapour. The character of the coupling is obtained on the basis of an appropriate, and rather general, matching condition. If we turn to the nonlinear theory, then, particularly because of the healing terms, it can be described only in outline, but this is sufficient to indicate the relevance, perhaps not surprisingly, of the Burgers equation. (Various simple special cases were examined to check the sign of the diffusivity: it was positive in all cases considered.)

The point was made earlier that all relevant choices of $\delta = \delta(\epsilon)$ are essentially embedded in this single far-field theory. On the one hand this statement can be made precise by remembering that δ is always removable by a suitable scaling. Thus there exist time and distance scales for *arbitrary* δ, for which the Burgers equation is valid. We could equally argue that, if $\epsilon = o(\delta)$, then with $\mathcal{T} = \delta t$, $\xi = O(1)$ we obtain the heat-conduction equation, which has decaying solutions obviating the balance with nonlinearity. Of course, nonlinear terms are still relevant on longer time and distance scales. Similarly, if $\delta = o(\epsilon)$, then with $\mathcal{T} = \epsilon t$, $\xi = O(1)$ it is clear that we shall generate the unidirectional nonlinear wave equation (without diffusivity), but then the Burgers equation is valid on shorter lengthscales as the wave steepens. On the foregoing it seems eminently reasonable to describe the Burgers equation as the generic equation for the evolution of small-amplitude waves in third sound. Nevertheless, other far-field nonlinear theories are available by suitably choosing the ϵ-dependence of other parameters (see Johnson 1984). However, it is also clear that a far-field theory that ignores the viscous terms at the expense of, say, dispersive terms must be constructed with some care. This is simply because these viscous terms appear in the form $1/\alpha_2$, which suggests a non-uniformity as $\alpha_2 \rightarrow 0$ (the natural limit

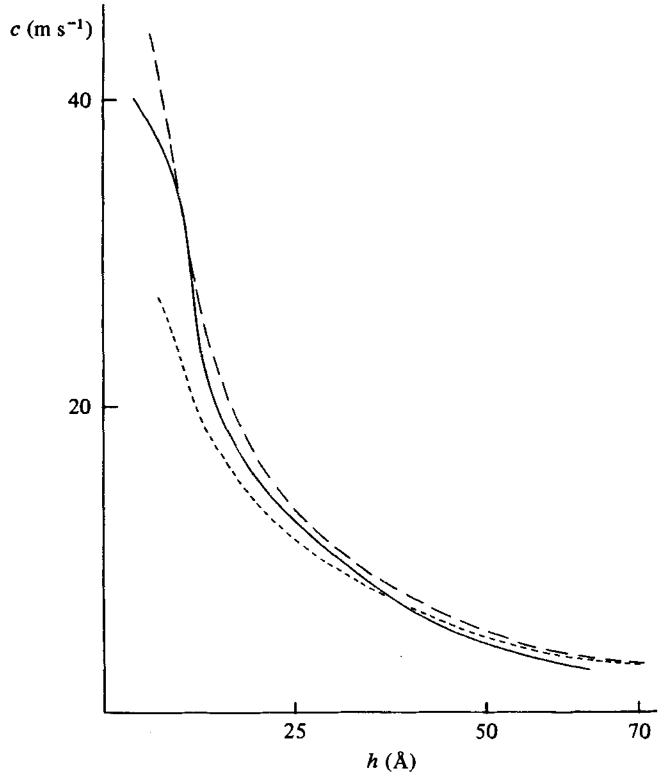

FIGURE 5. The propagation speed against film thickness at $T = 1.3$ K: estimate with healing (----), no healing (——); experimental (——) at $T = 1.315$ K.

to impose to remove viscous but retain other terms). These very particular difficulties are beyond the scope of the current study.

The numerical results based on the analytical work presented here fall naturally into one of two categories, and for two quite separate reasons. The first contains the equilibrium state of the liquid film in the absence of waves, and this aspect has been determined quite accurately. The second category includes the wave properties, of which only the wave speed has been estimated, and then only very crudely.

The effects of healing, which are dominant over any other when we compare our results with the 'classical' theory (Atkins & Rudnick 1970), are most evident in the equilibrium superfluid density. The typical density distributions (figure 2) show the distinct 'healing layers' near $y = 0, 1$ for the films of medium thickness. However, for very thin films (somewhat less than 15 Å, say) the maximum superfluid density is well below the bulk value, and then healing is significant throughout the layer. This is particularly clear in the superfluid-fraction versus film-thickness curves (figure 4). The occurrence of the zero solution for $\hat{\rho}_s(y)$ also manifests itself in this figure; this phenomenon is usually referred to as 'onset'.

Experimental studies with thin films (e.g. Chester & Yang 1973) have led to a better understanding of the onset region, showing in particular that there is no superfluid present below a critical thickness (which of course varies with temperature). Our calculations show that the superfluid fraction is certainly zero at 5 Å but not at 7.5 Å (for $T = 1.3$ K), and zero at 10 Å but not at 12.5 Å (for $T = 1.8$ K). A more precise estimate of the critical thickness was not undertaken owing to the immense amount

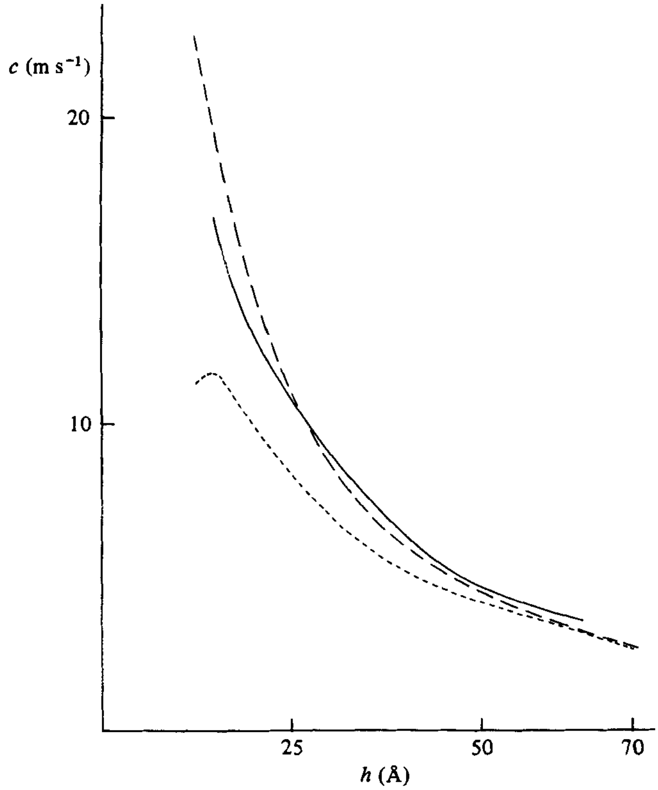

FIGURE 6. The propagation speed against film thickness at $T = 1.8$ K: estimate with healing (----), no healing (——); experimental (——) at $T = 1.778$ K.

of computer time involved in these integrations. From Chester & Yang (1973) the thickness of the layer at onset is 7.1 Å at 1.275°, 8.1 Å at 1.355°, and 10.8 Å at 1.808° (taking one atomic layer to be 3.6 Å). The computed values shown in figure 4 compare quite favourably with these experimental values (but note that the temperatures do not correspond precisely). This can be reasonably interpreted as a success for the continuum theory, with healing included.

The fact that equation (2*d*) for $\hat{\rho}_s(y)$ predicts the onset phenomenon is, itself, not at all surprising. This equation is akin to equations with elliptic-function solutions, for which it is well known that a zero solution is the only available one in certain circumstances. More specifically, it is reasonable to expect that $\hat{\rho}_s(y) \equiv 0$ if $\hat{\rho}_s(h) = \hat{\rho}_s(0) = 0$ and $h < h_c$ for some h_c. A particularly simple case can be tested directly by modelling the Helmholtz free energy as

$$a_0 + a_2(\hat{\rho}_s - \hat{\rho}_{s0})^2,$$

where a_0, a_2 and $\hat{\rho}_{s0}$ are constants (this latter being a measure of the bulk value of $\hat{\rho}_s$). Equation (2*d*) can now be integrated in terms of elliptic functions and the onset exhibited.

Another aspect of some note is that the agreement between the theoretical and experimental results is evidence for zero superfluid density both at the free surface and in the solid layer below the solid/liquid phase transition. The slight doubt concerning the specific boundary conditions to be used on $y = 0, 1$, and whether experimentalists measure any superfluid in the solid layer, have been areas of some uncertainty in recent years.

The total density distribution is shown in figure 3, and this is included essentially to demonstrate the effects of compressibility. These are particularly noticeable near the wall, where the van der Waals force is very high. Thus, typically, the compressibility of helium II can be regarded as significant in a restricted region (say 5–10 Å thick) in the neighbourhood of the wall. In other words, for films that are less than about 10 Å thick, we can expect that compressibility is an important factor throughout the layer, whereas for films in excess of 100 Å it is relatively unimportant.

The main thrust of the work presented here is to describe how the linear theory – and in outline the nonlinear – can be developed on the basis of fairly weak assumptions. However, it would be churlish to ignore any numerical estimates of the propagation speed c itself. As we have seen, the equilibrium state can be determined with some reasonable accuracy, although the same cannot be said of c, at least not with the data currently available to the author. The results shown in figures 5 and 6 are for the two temperatures under discussion (1.3 K, 1.8 K), and include the experimental curves quoted by Putterman (1974). Also included in the figures are the curves obtained from Johnson (1978*a*), which do not incorporate healing. Put at its simplest level it was expected that the effects of healing would slightly reduce the values of c, at least for the thinner films, and so bring them into line with the experimental results. The reduction is all too plain: our cavalier use of the updated superfluid density with the otherwise undoctored values from Johnson (1978*a*) is extreme. The tendency is in the right direction, and so it is to be hoped that all the additional terms available in our theory (particularly the effects of healing in the perturbed state) will alleviate – but not reverse – this trend.

In conclusion we address the question as to the efficacy of the continuum equations used here to describe third sound. It must be said that the evidence presented in the foregoing paragraphs does not constitute a conclusive case one way or the other. The equilibrium state has been represented with considerable success and the predictions of onset are very much an unlooked-for bonus. However, the modelling of third sound at this level of sophistication is still very much an open question. The introduction of healing, relaxation, etc. cannot be fully exploited until the numerical problems have been overcome, but this is not a difficulty in principle. It should not be long before a sufficiently detailed and accurate representation of the Helmholtz free energy is available. Nevertheless the continuum equations in general – and the modelling of third sound in particular – may immediately prove a useful tool in the study of simplified models for $A(\rho, \rho_s, T)$ for which the numerical work is straightforward.

The author would like to extend his thanks to Professor R. J. Donnelly, Dr R. N. Hills and Professor P. H. Roberts for their continued interest in this work and for making available their calculations on the thermodynamics of superfluid helium. The author is also very grateful to Dr C. A. Jones, of this department, for providing a suitable integration routine. The numerical work was performed on the IBM 370/168 machine at the University of Newcastle upon Tyne.

Appendix A

Here we produce the governing equations and boundary conditions written in non-dimensional variables.

Equations for the liquid

Mass:

$$\frac{\partial r}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\frac{\partial}{\partial y}[v_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]=0. \qquad (\mathrm{A}\ 1)$$

x-momentum:

$$\begin{aligned}&\frac{\partial}{\partial t}[u_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\epsilon\frac{\partial}{\partial x}[u_{\mathrm{n}}^2(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+u_{\mathrm{s}}^2(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\\ &\quad+\epsilon\frac{\partial}{\partial y}[u_{\mathrm{n}}v_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+u_{\mathrm{s}}v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\\ &\quad=-\frac{\partial p}{\partial x}-\epsilon\delta^2\beta\frac{\partial}{\partial x}\left[\frac{(\partial r_{\mathrm{s}}/\partial x)^2}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\right]-\beta\frac{\partial}{\partial y}\left[\frac{(\partial r_{\mathrm{s}}/\partial x)\,(\mathrm{d}\hat{\rho}_{\mathrm{s}}/\mathrm{d}y+\epsilon\,\partial r_{\mathrm{s}}/\partial y)}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\right]\\ &\quad+\delta(\alpha_1+\alpha_2)\frac{\partial^2 u_{\mathrm{n}}}{\partial x^2}+\frac{1}{2}\frac{\alpha^2}{\delta}\frac{\partial^2 u_{\mathrm{n}}}{\partial y^2}+\delta(\alpha_1+\tfrac{1}{2}\alpha_2)\frac{\partial^2 v_{\mathrm{n}}}{\partial x\,\partial y}.\end{aligned} \qquad (\mathrm{A}\ 2)$$

y-momentum:

$$\begin{aligned}&\epsilon\delta^2\left\{\frac{\partial}{\partial t}[v_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\epsilon\frac{\partial}{\partial x}[u_{\mathrm{n}}v_{\mathrm{n}}(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+u_{\mathrm{s}}v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right.\\ &\quad\left.+\epsilon\frac{\partial}{\partial y}[v_{\mathrm{n}}^2(\hat{\rho}_{\mathrm{n}}+\epsilon r_{\mathrm{n}})+v_{\mathrm{s}}^2(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right\}\\ &=-\frac{\mathrm{d}\hat{p}}{\mathrm{d}y}-\epsilon\frac{\partial p}{\partial y}-(\hat{\rho}+\epsilon r)\frac{\mathrm{d}\omega}{\mathrm{d}y}-\epsilon\delta^2\beta\frac{\partial}{\partial x}\left[\frac{(\partial r_{\mathrm{s}}/\partial x)\,(\mathrm{d}\hat{\rho}_{\mathrm{s}}/\mathrm{d}y+\epsilon\,\partial r_{\mathrm{s}}/\partial y)}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\right]\\ &\quad-\beta\frac{\partial}{\partial y}\left[\frac{(\mathrm{d}\hat{\rho}_{\mathrm{s}}/\mathrm{d}y+\epsilon\partial r_{\mathrm{s}}/\partial y)^2}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\right]+\epsilon\delta(\alpha_1+\alpha_2)\frac{\partial^2 v_{\mathrm{n}}}{\partial y^2}\\ &\qquad\qquad+\tfrac{1}{2}\epsilon\delta^3\alpha_2\frac{\partial^2 v_{\mathrm{n}}}{\partial x^2}+\epsilon\delta(\alpha_1+\tfrac{1}{2}\alpha_2)\frac{\partial^2 u_{\mathrm{n}}}{\partial x\,\partial y}.\end{aligned} \qquad (\mathrm{A}\ 3)$$

Irrotationality:

$$\begin{aligned}&\left(a+\rho\frac{\partial a}{\partial \rho}\right)_{\substack{\hat{\rho}+\epsilon r\\ \hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}\\ 1+\delta\epsilon\tau}}+\omega+\epsilon\frac{\partial \phi_{\mathrm{s}}}{\partial t}+\tfrac{1}{2}\epsilon^2[u_{\mathrm{s}}^2+\delta^2 v_{\mathrm{s}}^2-(u_{\mathrm{n}}-u_{\mathrm{s}})^2-\delta^2(v_{\mathrm{n}}-v_{\mathrm{s}})^2]\\ &\quad-\frac{\epsilon\delta\gamma}{(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})^{\frac{1}{2}}}\left\{\frac{\partial r_{\mathrm{s}}}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\frac{\partial}{\partial y}[v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right\}=0.\end{aligned} \qquad (\mathrm{A}\ 4)$$

State:

$$\hat{p}+\epsilon p=(\hat{\rho}+\epsilon r)^2\frac{\partial a}{\partial \rho}+(\hat{\rho}+\epsilon r)\,(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})\frac{\partial a}{\partial \rho_{\mathrm{s}}}-\epsilon\delta^2\beta\frac{\partial^2 r_{\mathrm{s}}}{\partial x^2}-\beta\left(\frac{\mathrm{d}^2\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y^2}+\epsilon\frac{\partial^2 r_{\mathrm{s}}}{\partial y^2}\right). \qquad (\mathrm{A}\ 5)$$

Constitutive:

$$\begin{aligned}&(\hat{\rho}+\epsilon r)\frac{\partial a}{\partial \rho_{\mathrm{s}}}+\tfrac{1}{2}\epsilon^2[(u_{\mathrm{n}}-u_{\mathrm{s}})^2+\delta^2(v_{\mathrm{n}}-v_{\mathrm{s}})^2]-\frac{\beta}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\left[\frac{\mathrm{d}^2\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y^2}+\epsilon\frac{\partial^2 r_{\mathrm{s}}}{\partial y^2}+\epsilon\delta^2\frac{\partial^2 r_{\mathrm{s}}}{\partial x^2}\right]\\ &\quad+\frac{1}{2}\frac{\beta}{(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})^2}\left[\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}+\epsilon\frac{\partial r_{\mathrm{s}}}{\partial y}\right)^2+\epsilon^2\delta^2\left(\frac{\partial r_{\mathrm{s}}}{\partial x}\right)^2\right]\\ &\quad+\frac{\epsilon\delta\gamma}{(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})^{\frac{1}{2}}}\left\{\frac{\partial r_{\mathrm{s}}}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\frac{\partial}{\partial y}[v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right\}=0.\end{aligned} \qquad (\mathrm{A}\ 6)$$

Energy:

$$\begin{aligned}(1+\delta\epsilon\tau)\left\{\frac{\partial}{\partial t}\left[(\hat{\rho}+\epsilon r)\frac{\partial a}{\partial T}\right]+\epsilon\frac{\partial}{\partial x}\left[u_{\mathrm{n}}(\hat{\rho}+\epsilon r)\frac{\partial a}{\partial T}\right]+\epsilon\frac{\partial}{\partial y}\left[v_{\mathrm{n}}(\hat{\rho}+\epsilon r)\frac{\partial a}{\partial T}\right]\right\}\\ +\frac{\epsilon^2\delta\gamma}{(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})^{\frac{1}{2}}}\left\{\frac{\partial r_{\mathrm{s}}}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\frac{\partial}{\partial y}[v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right\}^2\\ +\epsilon^2\delta\alpha_1\left(\frac{\partial u_{\mathrm{n}}}{\partial x}+\frac{\partial v_{\mathrm{n}}}{\partial y}\right)^2+\epsilon^2\delta\alpha_2\left[\left(\frac{\partial u_{\mathrm{n}}}{\partial x}\right)^2+\left(\frac{\partial v_{\mathrm{n}}}{\partial y}\right)^2\right]\\ +\frac{1}{2}\frac{\epsilon^2}{\delta}\alpha_2\left(\frac{\partial u_{\mathrm{n}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{n}}}{\partial x}\right)^2+\epsilon\delta^2\kappa\left(\frac{\partial^2\tau}{\partial x^2}+\frac{1}{\delta^2}\frac{\partial^2\tau}{\partial y^2}\right)=0.\end{aligned} \tag{A 7}$$

Boundary conditions on $y=0$:

$$u_{\mathrm{n}}=v_n=v_{\mathrm{s}}=0 \quad \text{on} \quad y=0, \tag{A 8}$$

$$\partial\tau/\partial y=0 \quad \text{on} \quad y=0. \tag{A 9}$$

Equations for the vapour

Mass:

$$\frac{1}{\varDelta}\frac{\partial r_{\mathrm{g}}}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]+\frac{\partial}{\partial y}[v_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]=0. \tag{A 10}$$

x-momentum:

$$\begin{aligned}\frac{1}{\varDelta}\frac{\partial}{\partial t}[u_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]+\epsilon\frac{\partial}{\partial x}[u_{\mathrm{g}}^2(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]+\epsilon\frac{\partial}{\partial y}[u_{\mathrm{g}}v_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]\\ =-\frac{\partial p_{\mathrm{g}}}{\partial x}+\delta(\alpha_{\mathrm{g1}}+\alpha_{\mathrm{g2}})\frac{\partial^2 u_{\mathrm{g}}}{\partial x^2}+\frac{1}{2}\frac{\alpha_{2\mathrm{g}}}{\delta}\frac{\partial^2 u_{\mathrm{g}}}{\partial y^2}+\delta(\alpha_{1\mathrm{g}}+\tfrac{1}{2}\alpha_{2\mathrm{g}})\frac{\partial^2 v_{\mathrm{g}}}{\partial x\,\partial y}.\end{aligned} \tag{A 11}$$

y-momentum:

$$\begin{aligned}\epsilon\delta^2\left\{\frac{1}{\varDelta}\frac{\partial}{\partial t}[v_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]+\epsilon\frac{\partial}{\partial x}[u_{\mathrm{g}}v_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]+\epsilon\frac{\partial}{\partial y}[v_{\mathrm{g}}^2(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})]\right\}\\ =-\frac{\mathrm{d}\hat{p}_{\mathrm{g}}}{\mathrm{d}y}-\epsilon\frac{\partial p_{\mathrm{g}}}{\partial y}-(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})\frac{1}{\varDelta^2}\frac{\mathrm{d}\omega}{\mathrm{d}y}+\epsilon\delta(\alpha_{\mathrm{g1}}+\alpha_{\mathrm{g2}})\frac{\partial^2 v_{\mathrm{g}}}{\partial y^2}+\tfrac{1}{2}\epsilon\delta^3\alpha_{\mathrm{g2}}\frac{\partial^2 v_{\mathrm{g}}}{\partial x^2}\\ +\epsilon\delta(\alpha_{\mathrm{g1}}+\tfrac{1}{2}\alpha_{\mathrm{g2}})\frac{\partial^2 u_{\mathrm{g}}}{\partial x\,\partial y}.\end{aligned} \tag{A 12}$$

State:

$$\hat{p}_{\mathrm{g}}+\epsilon p_{\mathrm{g}}=(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})^2\frac{\partial a_{\mathrm{g}}}{\partial \rho}. \tag{A 13}$$

Energy

$$\begin{aligned}(1+\epsilon\delta\tau_{\mathrm{g}})\left\{\frac{1}{\varDelta}\frac{\partial}{\partial t}\left[(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})\frac{\partial a_{\mathrm{g}}}{\partial T}\right]+\epsilon\frac{\partial}{\partial x}\left[u_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})\frac{\partial a_{\mathrm{g}}}{\partial T}\right]+\epsilon\frac{\partial}{\partial y}\left[v_{\mathrm{g}}(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})\frac{\partial a_{\mathrm{g}}}{\partial T}\right]\right\}\\ +\epsilon^2\delta\alpha_{\mathrm{g1}}\left(\frac{\partial u_{\mathrm{g}}}{\partial x}+\frac{\partial v_{\mathrm{g}}}{\partial y}\right)^2+\epsilon^2\delta\alpha_{\mathrm{g2}}\left[\left(\frac{\partial u_{\mathrm{g}}}{\partial x}\right)^2+\left(\frac{\partial v_{\mathrm{g}}}{\partial y}\right)^2\right]+\frac{1}{2}\frac{\epsilon^2}{\delta}\alpha_{\mathrm{g2}}\left(\frac{\partial u_{\mathrm{g}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{g}}}{\partial x}\right)^2\\ +\epsilon\kappa_{\mathrm{g}}\left(\delta^2\frac{\partial^2\tau_{\mathrm{g}}}{\partial x^2}+\frac{\partial^2\tau_{\mathrm{g}}}{\partial y^2}\right)=0.\end{aligned} \tag{A 14}$$

Interfacial conditions across $y = 1+\epsilon\eta$ $(\tan\theta = \epsilon\delta\,\partial\eta/\partial x)$

Mass:

$$\frac{j}{\cos\theta} = \epsilon\delta(\hat{\rho}+\epsilon r)\left(v_{\mathrm{n}}-\frac{\partial\eta}{\partial t}-\epsilon u_{\mathrm{n}}\frac{\partial\eta}{\partial x}\right) = \epsilon\delta\Delta\sigma(\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}})\left(v_{\mathrm{g}}-\frac{1}{\Delta}\frac{\partial\eta}{\partial t}-\epsilon u_{\mathrm{g}}\frac{\partial\eta}{\partial x}\right). \quad \text{(A 15)}$$

Normal stress:

$$\begin{aligned}&\hat{p}+\epsilon p-\sigma\Delta^2(\hat{p}_{\mathrm{g}}+\epsilon p_{\mathrm{g}})-\epsilon\delta\alpha_1\left(\frac{\partial u_{\mathrm{n}}}{\partial x}+\frac{\partial v_{\mathrm{n}}}{\partial y}\right)+\epsilon\delta\sigma\Delta^2\alpha_{\mathrm{g}1}\left(\frac{\partial u_{\mathrm{g}}}{\partial x}+\frac{\partial v_{\mathrm{g}}}{\partial y}\right)\\&+\epsilon\delta\cos^2\theta\left[\sigma\Delta^2\alpha_{\mathrm{g}2}\left\{\frac{\partial v_{\mathrm{g}}}{\partial y}+\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\frac{\partial u_{\mathrm{g}}}{\partial x}-\epsilon\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{g}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{g}}}{\partial x}\right)\right\}\right.\\&\left.-\alpha_2\left\{\frac{\partial v_{\mathrm{n}}}{\partial y}+\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\frac{\partial u_{\mathrm{n}}}{\partial x}-\epsilon\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{n}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{n}}}{\partial x}\right)\right\}\right]\\&-j^2\left(\frac{1}{\hat{\rho}+\epsilon r}-\frac{1}{\sigma}\frac{1}{\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}}}\right)+\left(\frac{\beta\cos^2\theta}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\right)\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}+\epsilon\frac{\partial r_{\mathrm{s}}}{\partial y}-\epsilon^2\delta^2\frac{\partial\eta}{\partial x}\frac{\partial r_{\mathrm{s}}}{\partial x}\right)^2 = 0. \quad \text{(A 16)}\end{aligned}$$

Tangential stress:

$$\begin{aligned}&\epsilon\left[\alpha_2\left\{2\epsilon\delta\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{n}}}{\partial x}-\frac{\partial v_{\mathrm{n}}}{\partial y}\right)-\left(1-\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\right)\left(\frac{\partial u_{\mathrm{n}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{n}}}{\partial x}\right)\right\}\right.\\&\left.-\Delta^2\sigma\alpha_{\mathrm{g}2}\left\{2\epsilon\delta\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{g}}}{\partial x}-\frac{\partial v_{\mathrm{g}}}{\partial y}\right)-\left(1-\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\right)\left(\frac{\partial u_{\mathrm{g}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{g}}}{\partial x}\right)\right\}\right]\\&+\frac{\epsilon\delta\beta}{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}\left[\frac{\partial r_{\mathrm{s}}}{\partial x}+\frac{\partial\eta}{\partial x}\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}+\epsilon\frac{\partial r_{\mathrm{s}}}{\partial y}\right)\right]\left(\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}+\epsilon\frac{\partial r_{\mathrm{s}}}{\mathrm{d}y}-\epsilon^2\delta^2\frac{\partial\eta}{\partial x}\frac{\partial r_{\mathrm{s}}}{\partial x}\right) = 0. \quad \text{(A 17)}\end{aligned}$$

Energy balance:

$$\left[\psi-(1+\epsilon\delta\tau)\left(\frac{\partial a}{\partial T}-\Delta^2\frac{\partial a_{\mathrm{g}}}{\partial T}\right)\right]\frac{j}{\cos\theta} = \epsilon\delta\left[\frac{\partial}{\partial y}(\kappa\tau-\Delta^3\sigma\kappa_{\mathrm{g}}\tau_{\mathrm{g}})-\epsilon\delta^2\frac{\partial\eta}{\partial x}\frac{\partial}{\partial x}(\kappa\tau-\Delta^3\sigma\kappa_{\mathrm{g}}\tau_{\mathrm{g}})\right]. \quad \text{(A 18)}$$

Difference in potential:

$$\begin{aligned}\psi = {}& a+\frac{\hat{p}+\epsilon p}{\hat{\rho}+\epsilon r}-\Delta^2\left(a_{\mathrm{g}}+\frac{\hat{p}_{\mathrm{g}}+\epsilon p_{\mathrm{g}}}{\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}}}\right)+\tfrac{1}{2}\epsilon^2[u_{\mathrm{n}}^2+\delta^2v_{\mathrm{n}}^2-\Delta^2(u_{\mathrm{g}}^2+\delta^2v_{\mathrm{g}}^2]\\&+\tfrac{1}{2}\epsilon^2\frac{\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}}}{\hat{\rho}+\epsilon r}[(u_{\mathrm{n}}-u_{\mathrm{s}})^2+\delta^2(v_{\mathrm{n}}-v_{\mathrm{s}})^2]-\epsilon\delta j\cos\theta\frac{\partial\eta}{\partial t}\left[\frac{1}{\hat{\rho}+\epsilon r}-\frac{1}{\sigma}\frac{1}{\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}}}\right]\\&+\frac{\epsilon\delta\gamma(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})^{\frac{1}{2}}}{\hat{\rho}+\epsilon r}\left\{\frac{1}{\Delta}\frac{\partial r_{\mathrm{s}}}{\partial t}+\frac{\partial}{\partial x}[u_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]+\frac{\partial}{\partial y}[v_{\mathrm{s}}(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})]\right\}\\&-\epsilon\delta\left[\frac{\alpha_1}{\hat{\rho}+\epsilon r}\left(\frac{\partial u_{\mathrm{n}}}{\partial x}+\frac{\partial v_{\mathrm{n}}}{\partial y}\right)-\frac{\Delta^2\alpha_{\mathrm{g}1}}{\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}}}\left(\frac{\partial u_{\mathrm{g}}}{\partial x}+\frac{\partial v_{\mathrm{g}}}{\partial y}\right)\right]\\&+\frac{\beta\cos^2\theta}{2(\hat{\rho}+\epsilon r)(\hat{\rho}_{\mathrm{s}}+\epsilon r_{\mathrm{s}})}\left[\frac{\mathrm{d}\hat{\rho}_{\mathrm{s}}}{\mathrm{d}y}+\epsilon\frac{\partial r_{\mathrm{s}}}{\partial y}-\epsilon^2\delta^2\frac{\partial\eta}{\partial x}\frac{\partial r_{\mathrm{s}}}{\partial x}\right]^2\\&-\epsilon\delta\cos^2\theta\left\{\frac{\alpha_2}{\hat{\rho}+\epsilon r}\left[\frac{\partial v_{\mathrm{n}}}{\partial y}+\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\frac{\partial u_{\mathrm{n}}}{\partial x}-\epsilon\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{n}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{n}}}{\partial x}\right)\right]\right.\\&\left.-\frac{\Delta^2\alpha_{\mathrm{g}2}}{\hat{\rho}_{\mathrm{g}}+\epsilon r_{\mathrm{g}}}\left[\frac{\partial v_{\mathrm{g}}}{\partial y}+\left(\epsilon\delta\frac{\partial\eta}{\partial x}\right)^2\frac{\partial u_{\mathrm{g}}}{\partial x}-\epsilon\frac{\partial\eta}{\partial x}\left(\frac{\partial u_{\mathrm{g}}}{\partial y}+\delta^2\frac{\partial v_{\mathrm{g}}}{\partial x}\right)\right]\right\}. \quad \text{(A 19)}\end{aligned}$$

Constitutive:

$$j = \nu\psi. \tag{A 20}$$

Continuity:

$$u_{\mathrm{n}} + \epsilon\delta^2 v_{\mathrm{n}} \frac{\partial\eta}{\partial x} = \varDelta\left(u_{\mathrm{g}} + \epsilon\delta^2 v_{\mathrm{g}} \frac{\partial\eta}{\partial x}\right), \quad \tau = \tau_{\mathrm{g}}. \tag{A 21}$$

Appendix B

The linear wave equation describing the propagation of third sound is expressed as

$$\frac{\partial^2\eta_0}{\partial t^2} - c^2\frac{\partial^2\eta_0}{\partial x^2} = 0,$$

with

$$c^2 = (l_\infty - \hat{a}_{T1})\int_0^1 \hat{\rho}_{\mathrm{s}}\,\mathrm{d}y \Big/ \left[\frac{\hat{\rho}_1 l_\infty + \sigma\hat{\rho}_{\mathrm{g}1}(l_1 - l_\infty) + \sigma n_0}{(-\lambda_0)} - \int_0^1 \{\hat{\rho}\hat{a}_{T\rho_{\mathrm{s}}} S_0 + (\hat{a}_T + \hat{\rho}\hat{a}_{T\rho} - \varDelta^2 a_{\mathrm{g}T\infty})\,R_0\}\,\mathrm{d}y\right],$$

where

$$n_0 = \int_1^\infty [(\hat{a}_{\mathrm{g}T} + \hat{\rho}_{\mathrm{g}}\,\hat{a}_{\mathrm{g}\rho T} - \hat{a}_{\mathrm{g}T\infty})\,R_{\mathrm{g}0} - \hat{\rho}_{\mathrm{g}\infty}\,\hat{a}_{\mathrm{g}\rho T\infty}\,R_{\mathrm{g}0\infty}]\,\mathrm{d}y + \hat{\rho}_{\mathrm{g}\infty}\,\hat{a}_{\mathrm{g}\rho T\infty}\,R_{\mathrm{g}0\infty}$$

and

$$\lambda_0 = \left.\frac{\tfrac{1}{2}\beta\sigma\hat{\rho}_{\mathrm{g}}(\hat{\rho}_{\mathrm{s}}'^2/\hat{\rho}_{\mathrm{s}})' - \hat{\rho}(\hat{\rho} - \sigma\hat{\rho}_{\mathrm{g}})\,\omega'}{(\hat{\rho} - \sigma\hat{\rho}_{\mathrm{g}})\,(\hat{\rho} + \beta\hat{\rho}_{\mathrm{s}}'' S_0/\hat{\rho}_{\mathrm{s}}) - \beta\hat{\rho}_{\mathrm{s}}'(\hat{\rho}\hat{\rho}_{\mathrm{s}} S_0' - \tfrac{1}{2}\sigma\hat{\rho}_{\mathrm{g}}\hat{\rho}_{\mathrm{s}}' S_0)/\hat{\rho}_{\mathrm{s}}^2}\right|_{y=1}$$

We have written $r_{\mathrm{g}0} = \eta_0\varDelta^{-2}R_{\mathrm{g}0}(y)$, and the subscripts 1 and ∞ denote evaluation on $y = 1, \infty$ respectively. The latent heats are defined by

$$l_1 = \hat{a}_{T1} - \varDelta^2\hat{a}_{\mathrm{g}T1}, \quad l_\infty = \hat{a}_{T1} - \varDelta^2\hat{a}_{\mathrm{g}T\infty}.$$

REFERENCES

Atkins, K. R. 1959 *Phys. Rev.* **113**, 962–965.

Atkins, K. R. & Rudnick, I. 1970 In *Progress in Low Temperature Physics*, vol. 7, pp. 37–76. North-Holland.

Bergman, D. 1969 *Phys. Rev.* **188**, 370–384.

Bergman, D. 1971 *Phys. Rev.* A **3**, 2058–2066.

Brooks, J. S. & Donnelly, R. J. 1977 *J. Phys. Chem. Ref. Data* **6**, 51–104.

Chester, M. & Yang, L. C. 1973 *Phys. Rev. Lett.* **31**, 1377–1380.

Everitt, C. W. F., Atkins, K. R. & Denenstein, A. 1962 *Phys. Rev. Lett.* **8**, 161–165.

Ginzburg, V. L. & Sobaynin, A. A. 1976 *Sov. Phys. Usp.* **19**, 773–812.

Hills, R. N. & Roberts, P. H. 1972 *J. Inst. Maths Applics* **9**, 56–67.

Hills, R. N. & Roberts, P. H. 1977 *Intl J. Engng Sci.* **15**, 305–316.

Hills, R. N. & Roberts, P. H. 1979 *J. Non-Equilib. Thermodyn.* **4**, 131–142.

Johnson, R. S. 1978*a* *Proc. R. Soc. Lond.* A **362**, 97–111.

Johnson, R. S. 1978*b* *Proc. R. Soc. Lond.* A **362**, 375–382.

Johnson, R. S. 1984 *J. Austral. Math. Soc.* B **25**, 311–348.

Khalatnikov, I. M. 1970 *Sov. Phys. JETP* **30**, 268–272.

Landau, L. D. 1941 *J. Phys.* **5**, 71–90.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics*. Pergamon.

Putterman, S. J. 1974 *Superfluid Hydrodynamics*. North-Holland.

Roberts, P. H. & Donnelly, R. J. 1974 *Ann. Rev. Fluid Mech.* **6**, 179–225.

Sobaynin, A. A. 1972 *Sov. Phys. JETP* **34**, 229–232.